Counting abelian varieties over finite fields
via Frobenius densities

Jeffrey D. Achter, S. Ali Altuğ, Luis Garcia and Julia Gordon
Appendix by Wen-Wei Li and Thomas Rüd

▪msp

Counting abelian varieties over finite fields via Frobenius densities

Jeffrey D. Achter, S. Ali Altuğ, Luis Garcia and Julia Gordon
Appendix by Wen-Wei Li and Thomas Rüd

Let $[X, \lambda]$ be a principally polarized abelian variety over a finite field with commutative endomorphism ring; further suppose that either X is ordinary or the field is prime. Motivated by an equidistribution heuristic, we introduce a factor $\nu_v([X, \lambda])$ for each place v of $\mathbb{Q}$, and show that the product of these factors essentially computes the size of the isogeny class of $[X, \lambda]$.

The derivation of this mass formula depends on a formula of Kottwitz and on analysis of measures on the group of symplectic similitudes and, in particular, does not rely on a calculation of class numbers.

1. Introduction

Let $[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ be a principally polarized g-dimensional abelian variety over the finite field $\mathbb{F}_q = \mathbb{F}_{p^e}$. Its isogeny class $I([X, \lambda], \mathbb{F}_q)$ is finite; our goal is to understand the (weighted by automorphism group) cardinality $\widetilde{\#}I([X, \lambda], \mathbb{F}_q)$.

A random matrix heuristic might suggest the following. Let $f_{X/\mathbb{F}_q}(T)$ be the characteristic polynomial of Frobenius of X. It is well known that $f_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$. Following Gekeler [2003], for a rational prime $\ell \nmid p \operatorname{disc}(f)$, one can define a number

$$\nu_\ell([X, \lambda], \mathbb{F}_q) = \lim_{n \to \infty} \frac{\#\{\gamma \in \mathrm{GSp}_{2g}(\mathbb{Z}_\ell/\ell^n) : \mathrm{charpoly}_\gamma(T) = f_{X/\mathbb{F}_q}(T) \bmod \ell^n\}}{\#\mathrm{GSp}_{2g}(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_{\mathrm{GSp}_{2g}}(\mathbb{Z}/\ell^n)}, \tag{1-1}$$

where GSp_{2g} is the group of symplectic similitudes of a symplectic space of dimension $2g$, and $\mathbb{A}_{\mathrm{GSp}_{2g}}$ is the space of characteristic polynomials of these similitudes.

For $\ell \nmid p \operatorname{disc}(f)$, in which case the conjugacy class is determined by the characteristic polynomial (see Lemma 3.1), we interpret $\nu_\ell[X, \lambda]$ as the deviation of the size of the conjugacy class with characteristic polynomial $f_{X/\mathbb{F}_q}(T)$ from the average size of a conjugacy class in $\mathrm{GSp}(\mathbb{Z}_\ell)$.

For $\ell \mid \operatorname{disc}(f_{X/\mathbb{F}_q}(T))$, since the characteristic polynomial need not determine a unique conjugacy class in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$, a slightly more involved definition of $\nu_\ell[X, \lambda]$ is needed; see (4-1). Similarly, we define quantities $\nu_p([X, \lambda], \mathbb{F}_q)$ and $\nu_\infty([X, \lambda], \mathbb{F}_q)$ using, respectively, equidistribution considerations for σ-conjugacy classes in $\mathrm{GSp}_{2g}(\mathbb{Q}_q)$ and the Sato–Tate measure on the compact form USp_{2g}.

Careless optimism might lead one to hope that $\widetilde{\#}I([X, \lambda], \mathbb{F}_q)$ is given by the product of the average archimedean and p-adic masses with the local deviations:

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_q) \propto \nu_\infty([X, \lambda], \mathbb{F}_q) \prod_\ell \nu_\ell([X, \lambda], \mathbb{F}_q). \tag{1-2}$$

This argument is (at best) superficially plausible. Nonetheless, in this paper we give a pure-thought proof of the following theorem:

Theorem A. *Let $[X, \lambda]$ be a principally polarized abelian variety over $\mathbb{F}_q$ with commutative endomorphism ring. Suppose that either X is ordinary or that $\mathbb{F}_q = \mathbb{F}_p$ is the prime field. Then*

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_q) = q^{\frac{1}{2} \dim(\mathcal{A}_g)} \tau_T \nu_\infty([X, \lambda], \mathbb{F}_q) \prod_\ell \nu_\ell([X, \lambda], \mathbb{F}_q). \tag{1-3}$$

Here $\dim(\mathcal{A}_g) = \frac{1}{2}g(g+1)$ and τ_T is the Tamagawa number of the algebraic torus associated with $[X, \lambda]$ in Section 2A.

As we have mentioned, this formulation is inspired by Gekeler [2003], who proves Theorem A for an ordinary elliptic curve E over a finite prime field $\mathbb{F}_p$. (In the case $g = 1$ considered by Gekeler, τ_T equals 1.) Roughly speaking, the strategy there is to compute the terms ν_ℓ explicitly, and show that the right-hand side of (1-3) actually computes, via Euler products, the value at $s = 1$ of a suitable L-function. One concludes via the analytic class number formula and the known description of the isogeny class $I(E, \mathbb{F}_q)$ as a torsor under the class group of the quadratic imaginary order attached to the Frobenius of E. This strategy was redeployed in [Achter and Williams 2015] and [Gerhard and Williams 2019] for certain ordinary abelian varieties.

More recently, Achter and Gordon [2017] showed directly that the right-hand side of (1-3) actually computes the product of the volume of a certain (adelic) quotient and an orbital integral on GL_2. Thanks to the work of Langlands [1973], and Dirichlet's class number formula, one has a direct proof that this product computes the size of the isogeny class of the elliptic curve.

In fact, Langlands' formula, originally developed to count points on modular curves over finite fields, has been generalized by Kottwitz [1992] to an essentially arbitrary Shimura variety of PEL type. We review this latter formula in Proposition 2.1, and refer to it as the Langlands–Kottwitz formula below. It comes as a product of an (adelic) volume of a torus and an orbital integral, this time over GSp_{2g}. Although the orbital integral in the Langlands–Kottwitz formula clearly decomposes as a product of local terms, the volume term, however, appears as a global quantity (a class number in the case of GL_2; see [Achter and Gordon 2017, Lemma A.4]). Thus an Euler product expression for $\widetilde{\#}I([X, \lambda], \mathbb{F}_q)$ such as the one in (1-3) is, at least, not immediate.

The content of the present paper is to prove that the Euler product given by the right-hand side of (1-3) is indeed equal to the product of the global volume and the orbital integral given by Kottwitz's formula. We establish this by a delicate analysis of the interplay between various measures on the relevant spaces.

This paper is the logical extension of [Achter and Gordon 2017], which worked out these details for the case with governing group $\mathrm{GSp}_2 = \mathrm{GL}_2$. The reader will correctly expect that the structure of the

argument is largely similar. However, the cohomological and combinatorial intricacies of symplectic similitude groups in comparison to general linear groups — in particular, the tori are much more complex and conjugacy and stable conjugacy need not coincide — mean that each stage is considerably more involved.

We highlight three particular issues that make the generalization from elliptic curves to higher rank not straightforward.

The first is already mentioned above — the difference between conjugacy and stable conjugacy in GSp_{2g} when $g > 1$. This issue is discussed in detail in Section 3, and leads to Definition 4.1, which (as we prove in Section 3) coincides with (1-1) when $\ell \nmid p \operatorname{disc}(f)$.

The second is the fundamental lemma for base change, which is used to relate a Gekeler-style ratio at p to the twisted orbital integral. The complicated function one generally gets as a result of base change is the reason we have to assume that X is ordinary if $q \neq p$; this is discussed in detail in Section 4C.

The last is that the tori in GSp_{2g} for $g \geq 2$ are significantly more complicated than those for $g = 1$. The global calculation in Section 5 reflects this complexity, and involves the Tamagawa number of the algebraic torus T. This number is well known to be 1 for $g = 1$, but for general g we have to leave it as an (unknown) constant; Thomas Rüd and (independently) Wen-Wei Li obtained suggestive partial results and kindly agreed to present them in the Appendix.

Perhaps not surprisingly, (1-3) can also be interpreted as a Smith–Minkowski–Siegel type mass formula (in the sense of Tamagawa–Weil) with explicit local masses (see [Gan and Yu 2000]). Here the underlying group, of course, is GSp_{2g} and the masses calculate sizes of the relevant isogeny classes. Although this point of view is interesting in its own right we do not pursue it further in this paper. We would, however, like to note that the appearance of Tamagawa numbers is natural in this context.

The present work is, of course, part of a long and thriving discourse on the size of isogeny classes of abelian varieties over finite fields. Deuring [1941] computes the size of an isogeny class of elliptic curves as a class number. Waterhouse [1969] reinterprets and extends Deuring's work to a much larger class of abelian varieties. The key point in his study is to consider separately the ℓ-primary components of the kernel of an isogeny, and model them using the various Tate modules (for ℓ prime to the characteristic) and the Dieudonné module. Both of these approaches have been revisited and enhanced in recent times. Yu and collaborators have undertaken a detailed analysis of, for example, the number of supersingular abelian varieties over finite fields; their answers are often expressed in terms of a mass formula which comes from a local lattice-theoretic perspective on the isogeny problem (e.g., [Xue and Yu 2021; Yu 2012]). In a somewhat different direction, Marseglia [2021] and Howe [2022] are often able to express the size of an isogeny class of principally polarized abelian varieties as a sum of suitably generalized class numbers.

In the approach taken here, the orbital integral in the Langlands–Kottwitz formula (Proposition 2.1) does the work of assimilating information from the different local lattice calculations. Class numbers necessarily arise from our formula, but are not built-in; see Section 6 for this emergence.

Notation. We work over a finite field $\mathbb{F}_q = \mathbb{F}_{p^e}$ of characteristic p. We will often drop the field from all

notation. We denote by $\mathbb{Q}_q$ the degree e unramified extension of $\mathbb{Q}_p$, and by $\mathbb{Z}_q$ its ring of integers; the p-th power automorphism of $\mathbb{F}_q$ lifts to the Frobenius automorphism σ of $\mathbb{Q}_q$.

Fix a positive integer g. Let $V = \mathbb{Z}^{\oplus 2g}$, endowed with basis $x_1, \ldots, x_g, y_1, \ldots, y_g$, and equip it with the symplectic form such that $\langle x_i, y_j \rangle = \delta_{ij}$ and $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 0$. Let G be the group of similitudes $G = \mathrm{GSp}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{GSp}_{2g}$; it has dimension $2g^2 + g + 1$ and rank $r = g + 1$. Its derived group is $G^{\mathrm{der}} \cong \mathrm{Sp}_{2g}$, and $G/G^{\mathrm{der}} \cong \mathbb{G}_m$. Let $\eta : G \to \mathbb{G}_m$ be the corresponding surjection (the multiplier map). We write T_{spl} for the split torus of diagonal matrices in G.

If K is a field, $\alpha \in G(K)$, and $\Gamma \subseteq G(K)$, we let $^\Gamma\alpha = \{\beta^{-1}\alpha\beta : \beta \in \Gamma\}$ be the orbit of α under Γ. Since G has simply connected derived group, the stable conjugacy class, or stable orbit, of α is those elements of $G(K)$ which are conjugate to α as elements of $G(\bar{K})$ [Kottwitz 1982, p. 785].

For $\alpha \in G(\mathbb{Q}_q)$, the twisted, or σ-, conjugacy class of α is $\{\beta^{-1}\alpha\beta^\sigma : \beta \in G(\mathbb{Q}_q)\}$; and the stable twisted conjugacy class of α is $G(\mathbb{Q}_q) \cap \{\beta^{-1}\alpha\beta^\sigma : \beta \in G(\bar{\mathbb{Q}}_q)\}$.

For an element $\gamma \in G(\mathbb{Q}_\ell)$, where ℓ is an arbitrary prime, the Weyl discriminant of γ is denoted by $D(\gamma)$: $D(\gamma) = \prod_{\alpha \in \Phi}(1 - \alpha(\gamma))$, where the product is over all roots of G (see Section 6A1 for details).

2. Background

2A. *The Kottwitz formula.* The key formula we need is developed by Kottwitz [1992]. In fact, the special case we need is detailed in [Kottwitz 1990, Section 12]. By way of establishing necessary notation, we review the relevant part of this work here.

Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g. An isogeny between two principally polarized abelian varieties $[X, \lambda], [Y, \mu] \in \mathcal{A}_g(\mathbb{F}_q)$ is an isogeny $\phi : X \to Y$ such that $m\phi^*\mu = n \cdot \lambda$ for some nonzero integers m and n. The isogeny class $I([X, \lambda], \mathbb{F}_q)$ is the set of all principally polarized abelian varieties $[Y, \mu]/\mathbb{F}_q$ admitting such an isogeny (over $\mathbb{F}_q$), and its weighted cardinality is

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_q) = \sum_{[Y,\mu]\in I([X,\lambda],\mathbb{F}_q)} \frac{1}{\#\mathrm{Aut}(Y, \mu)}.$$

The abelian variety $X/\mathbb{F}_q$ admits a Frobenius endomorphism $\varpi_{X/\mathbb{F}_q}$, with characteristic polynomial $f_{X/\mathbb{F}_q}(T)$ of degree $2g$. (By [Tate 1966], this polynomial determines the isogeny class of X as an unpolarized abelian variety.)

For each $\ell \neq p$, $H^1(X_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell)$ (the dual of the Tate module) is a free $\mathbb{Z}_\ell$-module of rank $2g$, endowed with a symplectic pairing $\langle \cdot, \cdot \rangle_\lambda$ induced by the polarization. The Frobenius endomorphism $\varpi_{X/\mathbb{F}_q}$ induces an element $\gamma_{X/\mathbb{F}_q, \ell} \in \mathrm{GSp}(H^1(X_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell), \langle \cdot, \cdot \rangle_\lambda)$, and thus an element of $G(\mathbb{Z}_\ell)$, well-defined up to conjugacy. Moreover, there is an equality of characteristic polynomials $f_{\gamma_{X/\mathbb{F}_q, \ell}}(T) = f_{X/\mathbb{F}_q}(T)$. Simultaneously considering all finite primes $\ell \neq p$, we obtain an adelic similitude $\gamma_{[X,\lambda]} \in G(\mathbb{A}_f^p)$. (Alternatively one can, of course, directly consider the action of $\varpi_{X/\mathbb{F}_q}$ on $H^1(X_{\bar{\mathbb{F}}_q}, \widehat{\mathbb{Z}}^p) = \lim_{p\nmid n} H^1(X_{\bar{\mathbb{F}}_q}, \mathbb{Z}/n)$.)

Similarly, the crystalline cohomology group $H^1_{\mathrm{cris}}(X, \mathbb{Q}_q)$ is endowed with an integral structure $H^1_{\mathrm{cris}}(X, \mathbb{Z}_q)$ and a σ-linear endomorphism F. It determines, up to σ-conjugacy, an element $\delta_{X/\mathbb{F}_q}$ of $G(\mathbb{Q}_q)$ with multiplier $\eta(\delta_{X/\mathbb{F}_q}) = p$.

The e-th iterate of F is linear, and in fact F^e is the endomorphism of $H^1_{\mathrm{cris}}(X, \mathbb{Q}_q)$ induced by $\varpi_{X/\mathbb{F}_q}$.

Let $T_{[X,\lambda]}/\mathbb{Q}$ represent the automorphism group of $[X, \lambda]$ in the category of abelian varieties up to $\mathbb{Q}$-isogeny. Concretely, the polarization λ induces a (Rosati) involution ($\dagger$) on $\mathrm{End}(X) \otimes \mathbb{Q}$; and for each $\mathbb{Q}$-algebra R, we have

$$T_{[X,\lambda]}(R) = \{\alpha \in (\mathrm{End}(X) \otimes R)^\times : \alpha\alpha^{(\dagger)} \in R^\times\}.$$

By Tate's theorem [1966], for $\ell \neq p$, $T_{[X,\lambda]}(\mathbb{Q}_\ell)$ is isomorphic to $G_{\gamma_{X/\mathbb{F}_q,\ell}}(\mathbb{Q}_\ell)$, the centralizer of $\gamma_{X/\mathbb{F}_q,\ell}$, and $T_{[X,\lambda]}(\mathbb{Q}_q)$ is isomorphic to $G_{\delta_{X/\mathbb{F}_q}\sigma}(\mathbb{Q}_p)$, the twisted centralizer of $\delta_{X/\mathbb{F}_q}$ in $G(\mathbb{Q}_q)$.

A direct analysis of the effect of isogenies on the first cohomology groups of abelian varieties shows:

Proposition 2.1 [Kottwitz 1990]. *The weighted cardinality of the isogeny class of* $[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ *is*

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_q) = \mathrm{vol}(T_{[X,\lambda]}(\mathbb{Q}) \backslash T_{[X,\lambda]}(\mathbb{A}_f)) \cdot \int_{G_{\gamma_{X/\mathbb{F}_q}}(\mathbb{A}_f^p) \backslash G(\mathbb{A}_f^p)} \mathbb{1}_{G(\widehat{\mathbb{Z}}_f^p)}(g^{-1}\gamma_{X/\mathbb{F}_q}g)\, dg$$

$$\cdot \int_{G_{\delta_{X/\mathbb{F}_q}\sigma}(\mathbb{Q}_p) \backslash G(\mathbb{Q}_q)} \mathbb{1}_{G(\mathbb{Z}_q)\, \mathrm{diag}(p,\dots,p,1,\dots,1)G(\mathbb{Z}_q)}(h^{-1}\delta_{X/\mathbb{F}_q}h^\sigma)\, dh. \quad (2\text{-}1)$$

In the orbital and twisted orbital integrals in (2-1), we choose the Haar measures on G which assign volume 1 to $G(\widehat{\mathbb{Z}}^p)$ and to $G(\mathbb{Z}_q)$, respectively. The choice of measure on T does not matter here, as long as the same measure is used to calculate the global volume. We define the specific measure on T in Section 5. It coincides with the canonical measure at all but finitely many places.

This formula appears in [Kottwitz 1990, p. 205]; see also [Kottwitz 1992] for its generalization to a much larger class of PEL Shimura varieties. As in [Achter and Gordon 2017, 2.4], the weighted cardinality accounts for the fact that we have not introduced a rigidifying level structure, and thus our objects admit nontrivial, albeit finite, automorphism groups.

Remark 2.2. Using Honda–Tate theory, one can find $\gamma_{X/\mathbb{F}_q,0} \in G(\mathbb{Q})$, well-defined up to $G(\overline{\mathbb{Q}})$-conjugacy, such that $\gamma_{X/\mathbb{F}_q,0}$ and $\gamma_{X/\mathbb{F}_q,\ell}$ are conjugate in $G(\overline{\mathbb{Q}}_\ell)$; see Kottwitz [1990, p. 206; 1992, p. 422]. Similarly, $\gamma_{X/\mathbb{F}_q,0}$ and $\mathrm{N}\,\delta_{X/\mathbb{F}_q}$ are conjugate in $G(\overline{\mathbb{Q}}_q)$, where N denotes the norm map

$$G(\mathbb{Q}_q) \to G(\mathbb{Q}_q), \quad \alpha \mapsto \alpha\alpha^\sigma \cdots \alpha^{\sigma^{e-1}}.$$

In particular, the characteristic polynomial of $\gamma_{X/\mathbb{F}_q,0}$ is $f_{X/\mathbb{F}_q}(T)$. In fact, by adjusting $\delta_{X/\mathbb{F}_q}$ in its twisted conjugacy class, we henceforth can and will assume that

$$\mathrm{N}(\delta_{X/\mathbb{F}_q}) \in G(\mathbb{Q}_p) \subset G(\mathbb{Q}_q) \quad (2\text{-}2)$$

[Kottwitz 1982, p. 206]. Then the group variety $T_{[X,\lambda]/\mathbb{F}_q}$ is isomorphic to the centralizer of $\mathrm{N}(\gamma_{X/\mathbb{F}_q,0})$ in G.

It turns out that, moreover, one can find a rational element $\gamma_0 \in G(\mathbb{Q})$ such that γ_0 is $G(\mathbb{Q}_\ell)$-conjugate to $\gamma_{X/\mathbb{F}_q,\ell}$ for every $\ell \neq p$ (see [Kisin 2017, p. 889]). Consequently, in (2-1) we could replace $\gamma_{X/\mathbb{F}_q}$ with a global object γ_0; but we will never use this fact in this paper.

In the remainder of this paper we fix a principally polarized abelian variety $[X, \lambda]/\mathbb{F}_q$ with commutative endomorphism ring $\mathrm{End}(X)$. (For example, any simple, ordinary abelian variety necessarily has a

commutative endomorphism ring [Waterhouse 1969, Theorem 7.2].) By Tate's theorem, the commutativity of $\text{End}(X)$ is equivalent to the condition that $T_{[X,\lambda]}$ is a maximal torus in G.

To ease notation slightly, we will write δ_0 and T for $\delta_{X/\mathbb{F}_q}$ and $T_{[X,\lambda]}$, respectively. If ℓ is a fixed, notationally suppressed prime, we will sometimes write γ_0 for $\gamma_{X/\mathbb{F}_q,\ell}$; by Remark 2.2, one may equally well let γ_0 be the image of some choice γ_0 in $G(\mathbb{Q})$ (though we will not be using it).

2B. *Structure of the centralizer.* For future use, we record some information about the centralizer $T = T_{[X,\lambda]}$. Recall that X is a g-dimensional abelian variety with commutative endomorphism ring. Then T is a maximal torus in G, and $K := \text{End}(X)^0 = \text{End}(X) \otimes \mathbb{Q}$ is a CM-algebra of degree $2g$ over $\mathbb{Q}$. Then K is isomorphic to a direct sum $K \cong \bigoplus_{i=1}^t K_i$ of CM-fields, and the Rosati involution on $\text{End}(X)$ induces a positive involution $a \mapsto \bar{a}$ on K, which in turn restricts to complex conjugation on each component K_i. Let $K^+ \subset K$ be the subalgebra fixed by the positive involution. Then $K^+ \cong \bigoplus_{i=1}^t K_i^+$, where K_i^+ is the maximal totally real subfield of K_i, and $[K^+ : \mathbb{Q}] = g$.

In general, if L is a field and M/L is a finite étale algebra, let $\boldsymbol{R}_{M/L}$ be Weil's restriction of scalars functor. The norm map $\text{N}_{M/L}$ induces a map of tori $\boldsymbol{R}_{M/L}\mathbb{G}_m \to \mathbb{G}_m$, and the norm-one torus is the kernel of the map

$$1 \to \boldsymbol{R}_{M/L}^{(1)}\mathbb{G}_m \to \boldsymbol{R}_{M/L}\mathbb{G}_m \xrightarrow{\text{N}_{M/L}} \mathbb{G}_m \to 1.$$

With these preparations we have

$$T^{\text{der}} := T \cap G^{\text{der}} \cong \boldsymbol{R}_{K^+/\mathbb{Q}}\boldsymbol{R}_{K/K^+}^{(1)}\mathbb{G}_m,$$

and T sits in the diagram

$$(2\text{-}3)$$

On points, we have

$$T(\mathbb{Q}) = \{a \in K^\times : a\bar{a} \in \mathbb{Q}^\times\} = \left\{(a_1, \ldots, a_t) \in \bigoplus K_i^\times : \text{there exists } c \in \mathbb{Q}^\times \text{ such that } a_i\bar{a}_i = c\right\},$$

$$T^{\text{der}}(\mathbb{Q}) = \{a \in K^\times : a\bar{a} = 1\} = \left\{(a_1, \ldots, a_t) \in \bigoplus K_i^\times : a_i\bar{a}_i = 1\right\}.$$

Let $\widetilde{T} = T^{\text{der}} \times \mathbb{G}_m$. It is not hard to write down an explicit isogeny $\alpha : \widetilde{T} \to T$ and a complementary isogeny $\beta : T \to \widetilde{T}$ such that $\alpha \circ \beta$ is the squaring map. We choose the maps which, on points, are given by

$$T \xrightarrow{\beta} \widetilde{T} \xrightarrow{\alpha} T, \qquad a \xrightarrow{\beta} (a\bar{a}^{-1}, a\bar{a}), \quad (b,c) \xrightarrow{\alpha} bc.$$

2C. *The Steinberg quotient.* Recall that we have fixed a maximal split torus T_{spl} in G; let W be the Weyl group of G relative to T_{spl}. Let $T_{\text{spl}}^{\text{der}} = T_{\text{spl}} \cap G^{\text{der}}$, and let $A^{\text{der}} = T_{\text{spl}}^{\text{der}}/W$ be the Steinberg quotient for the semisimple group G^{der}. It is isomorphic to the affine space of dimension $r - 1 = g$.

We let $\mathbb{A}_G = A^{\mathrm{der}} \times \mathbb{G}_m$ be the analogue of the Steinberg quotient for the reductive group G, and define a map

$$G \xrightarrow{\mathfrak{c}} \mathbb{A}_G, \quad \gamma \mapsto \left(\mathrm{tr}(\gamma), \mathrm{tr}(\wedge^2 \gamma), \ldots, \mathrm{tr}(\wedge^g \gamma), \eta(\gamma)\right). \tag{2-4}$$

Note that $\eta(\gamma) = \mathrm{tr}\left(\wedge^{g+1}(\gamma)\right)/\mathrm{tr}(\gamma)$; and if $\gamma \in G^{\mathrm{der}} \subset G$, then $\mathfrak{c}(\gamma) = (\mathfrak{c}^{\mathrm{der}}(\gamma), 1)$, where $\mathfrak{c}^{\mathrm{der}}$ is the usual Steinberg map.

2D. _Truncations._ Let ℓ be any finite prime (including $\ell = p$). Let $\pi_n = \pi_{\ell,n} : \mathbb{Z}_\ell \to \mathbb{Z}_\ell/\ell^n$ be the truncation map. For any $\mathbb{Z}_\ell$-scheme $\mathcal{X}$, we denote by $\pi_n^{\mathcal{X}}$ the corresponding map

$$\pi_n^{\mathcal{X}} : \mathcal{X}(\mathbb{Z}_\ell) \to \mathcal{X}(\mathbb{Z}_\ell/\ell^n)$$

induced by π_n. Given $S_n \subset \mathcal{X}(\mathbb{Z}_\ell/\ell^n)$, we will often set

$$\widetilde{S}_n = \pi_n^{-1}(S_n).$$

The projection maps π_n^G extend to a somewhat larger set of similitudes. Let $M(\mathbb{Z}_\ell)$ be the set of symplectic similitudes which stabilize the lattice $V \otimes \mathbb{Z}_\ell$. Then

$$M(\mathbb{Z}_\ell) = \mathrm{GSp}(V \otimes \mathbb{Q}_\ell) \cap \mathrm{End}(V \otimes \mathbb{Z}_\ell) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell) \cap \mathrm{Mat}_{2g}(\mathbb{Z}_\ell).$$

Inside this set, for each $d \geq 0$ we identify a subset

$$M(\mathbb{Z}_\ell)_d = \{A \in M(\mathbb{Z}_\ell) : \mathrm{ord}_\ell \det(A) \leq d\}.$$

Finally, let us denote by $M(\mathbb{Z}_\ell/\ell^n)_d$ the set

$$M(\mathbb{Z}_\ell/\ell^n)_d = \{A \in M(\mathbb{Z}_\ell/\ell^n) : \mathrm{ord}_\ell \det(A) \leq d\}.$$

Note that $M(\mathbb{Z}_\ell)_0 = G(\mathbb{Z}_\ell)$, and in the last definition, the condition on the determinant is not vacuous even if $d \gg n$, because it rules out the matrices of determinant zero.

With a certain amount of abuse, we introduce the following notion of "$M(\mathbb{Z}_\ell)_d$-conjugacy":

Definition 2.3. If $\gamma \in M(\mathbb{Z}_\ell)$ and in particular if $\gamma \in G(\mathbb{Z}_\ell)$, we will write $\gamma \sim_{M(\mathbb{Z}_\ell)_d} \gamma_0$ if there exists some $A \in M(\mathbb{Z}_\ell)_d$ such that $A\gamma = \gamma_0 A$.

Similarly, if $\bar{\gamma} \in M(\mathbb{Z}_\ell/\ell^n)$, we write $\bar{\gamma} \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0$ if there exists some $\bar{A} \in M(\mathbb{Z}_\ell/\ell^n)_d$ such that $\bar{A}\gamma = \pi_n(\gamma_0)\bar{A}$.

When n is small relative to d, truncations of $M(\mathbb{Z}_\ell)_d$- conjugate elements might not be $M(\mathbb{Z}_\ell/\ell^n)_d$-conjugate (since, e.g., all the elements $A \in M(\mathbb{Z}_\ell)$ satisfying $A\gamma = \gamma_0 A$ might project to 0 mod ℓ^n). Of course, this does not happen when $n \gg d$. We also note that trivially, if $\gamma \sim_{M(\mathbb{Z}_\ell)_{d_0}} \gamma_0$ for some d_0, then $\gamma \sim_{M(\mathbb{Z}_\ell)_d} \gamma_0$ for all $d \geq d_0$. The analogous statement holds for $\bar{\gamma} \in G(\mathbb{Z}_\ell/\ell^n)$ as long as $n \gg d$.

2E. _Measures and integrals._ As in [Achter and Gordon 2017], we need to explicitly work out the relationships between several different natural measures on the ℓ-adic points of varieties, especially groups and group orbits. The definitions introduced in [Achter and Gordon 2017, §3] (where a little more historical perspective is briefly reviewed) go through with minimal changes. We recall the relevant notation here.

<u>Serre–Oesterlé measure</u>: Serre [1981, §3], observed that for a smooth p-adic submanifold Y of $\mathbb{Z}_p^m$ of dimension d, there is a limit $\lim_{n\to\infty}|Y_n|p^{-nd}$, where Y_n is the reduction of Y modulo p^n (in our notation, $Y_n = \pi_n(Y)$). Moreover, Serre pointed out that this limit can be understood as the *volume* of Y with respect to a certain measure, which is canonical. The definition of this measure for more general sets Y was elaborated on by Oesterlé [1982] and by Veys [1992]. We refer to this measure as the Serre–Oesterlé measure, and denote it by μ^{SO}.

<u>Measures on groups</u>: Once and for all, we fix the measure $|dx|_\ell$ on the affine line $\mathbb{A}^1_{\mathbb{Q}_\ell}$ to be the translation-invariant measure such that $\mathrm{vol}_{|dx|_\ell}(\mathbb{Z}_\ell) = 1$. Then there are two fundamentally different approaches to defining measure. The first is, for any smooth algebraic variety $\mathcal{X}$ over $\mathbb{Q}_\ell$ with a nonvanishing top degree differential form ω on it, one gets the associated measure $|d\omega|_\ell$ on $\mathcal{X}(\mathbb{Q}_\ell)$. In particular, for a reductive group G, there is a canonical differential form ω_G, defined in the greatest generality by Gross [1997]. This gives a canonical measure $|d\omega_G|_\ell$ on $G(\mathbb{Q}_\ell)$. When G is split over $\mathbb{Q}$, this measure has an alternative description using point-counting over the finite field (i.e., it coincides with the Serre–Oesterlé measure μ_G^{SO} defined above):

$$\int_{G(\mathbb{Z}_\ell)} |d\omega_G|_\ell = \frac{\#G(\mathbb{Z}/\ell)}{\ell^{\dim(G)}}. \tag{2-5}$$

This observation is originally due to A. Weil [1982], and is actually built into his definition of integration on adeles. Weil's classical observation is precisely what makes this paper possible.

For groups, there is a second approach. Start with a Haar measure and normalize it so that some given maximal subgroup has volume 1. The choice of a "canonical" compact subgroup in this approach could lead to very interesting considerations (and is one of the main points of [Gross 1997]), but in our situation only one easy case is needed. For $G(\mathbb{Q}_\ell)$, the relevant maximal subgroup is $G(\mathbb{Z}_\ell)$; we denote such a Haar measure on $G(\mathbb{Q}_\ell)$ by μ_G^{can}.

<u>Geometric measure on orbits</u>: This is a measure constructed in [Frenkel et al. 2010] on a fibre of the Steinberg map $\mathfrak{c}\colon G \to \mathbb{A}_G$. Let ω_G be a volume form on G, and let ω_A be the volume form $\bigwedge dx_i \wedge (dx/|x|)$ on $\mathbb{A}_G \cong \mathbb{A}^{\mathrm{rank}(G)-1} \times \mathbb{G}_m$. On the fibre $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$, factor ω_G as

$$\omega_G = \omega_{\mathfrak{c}(\gamma)}^{\mathrm{geom}} \wedge \omega_A;$$

integrating $|\omega_{\mathfrak{c}(\gamma)}^{\mathrm{geom}}|$ defines a measure μ^{geom} on $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$.

Suppose ϕ is a locally constant compactly supported function on $G(\mathbb{Q}_\ell)$. Recall the family $\gamma_{X/\mathbb{F}_q,\ell}$ (and δ_0), whose centralizers are the sets of $\mathbb{Q}_\ell$-points of the algebraic torus $T := T_{[X,\lambda]}$. We use two different measures on the orbit ${}^{G(\mathbb{Q}_\ell)}\gamma_{X/\mathbb{F}_q,\ell} \cong T_{[X,\lambda]}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)$ to define an integral. When ℓ is fixed, we will often denote the element $\gamma_{X/\mathbb{F}_q,\ell}$ by γ_0; we define $\mu_{\gamma_0}^{\mathrm{Tama}}$ as the quotient measure $\mu_G^{\mathrm{can}}/\mu_T^{\mathrm{Tama}}$, where μ_T^{Tama} is the Tamagawa measure on T defined below in Section 5A, and let $\mu_{\gamma_0}^{\mathrm{geom}}$ be the geometric measure reviewed above. (Since the orbit of γ_0 is an open subset of $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$, the restriction of the geometric measure from $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ to the orbit makes sense.) Then for $\bullet \in \{\mathrm{Tama}, \mathrm{geom}\}$, set

$$O_{\gamma_0}^\bullet := \int_{T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \phi(g^{-1}\gamma_0 g) \, d\mu_{\gamma_0}^\bullet.$$

3. Conjugacy

3A. *Integral conjugacy.* To relate the right-hand side of (2-1) to the ratios ν_ℓ of (1-1), we interpret the orbital integral as the volume of the intersection of the $G(\mathbb{Q}_\ell)$-orbit of γ_0 with $G(\mathbb{Z}_\ell)$. For almost all ℓ, $G(\mathbb{Z}_\ell) \cap {}^{G(\mathbb{Q}_\ell)} \gamma_0 = {}^{G(\mathbb{Z}_\ell)} \gamma_0$:

Lemma 3.1. *Suppose* $\gamma_0 \in G(\mathbb{Z}_\ell)$ *and* $\ell \nmid D(\gamma_0)$. *If* $\gamma \in G(\mathbb{Z}_\ell)$, *then*

$$\gamma \sim_{G(\mathbb{Q}_\ell)} \gamma_0 \iff \gamma \sim_{G(\mathbb{Z}_\ell)} \gamma_0.$$

Proof. The hypothesis on γ_0 implies that the centralizer G_{γ_0} is a smooth torus over $\mathbb{Z}_\ell$, and thus the transporter from G_γ to G_{γ_0} is smooth over $\mathbb{Z}_\ell$ (e.g., [Conrad 2014, Proposition 2.1.2]).

Since γ and γ_0 are conjugate in $G(\mathbb{Q}_\ell)$, they have the same characteristic polynomial, and thus their reductions $\overline{\gamma}_0 = \pi_1^G(\gamma_0)$ and $\overline{\gamma} = \pi_1^G(\gamma)$ are stably conjugate in $G(\mathbb{Z}_\ell/\ell)$. By Lang's theorem, $\overline{\gamma}_0$ and $\overline{\gamma}$ are conjugate in $G(\mathbb{Z}_\ell/\ell)$; by smoothness of the transporter scheme, γ and γ_0 are conjugate in $G(\mathbb{Z}_\ell)$. □

If $\overline{\gamma}_0$ is *not* regular, then the set $G(\mathbb{Z}_\ell) \cap {}^{G(\mathbb{Q}_\ell)} \gamma_0$ generally consists of several different $G(\mathbb{Z}_\ell)$-orbits. Nonetheless, the number of distinct orbits is bounded; and membership in ${}^{G(\mathbb{Q}_\ell)}\gamma_0$ can be detected at a finite truncation level.

Lemma 3.2. *Suppose* $\gamma_0 \in G(\mathbb{Z}_\ell)$ *is regular semisimple. There exists an integer* $e = e(\gamma_0)$ *such that, if* $n \gg 0$ *and* $d > e$, *then for* $\gamma \in G(\mathbb{Z}_\ell/\ell^n)$, *the following conditions are equivalent*:

(1) $\gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0 \bmod \ell^n$, *and*

(2) *there exists some* $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ *such that* $\widetilde{\gamma} \bmod \ell^n = \gamma$ *and* $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$.

The statement is also true with $G(\mathbb{Z}_\ell)$ *replaced with* $M(\mathbb{Z}_\ell)$ *everywhere.*

Proof. We prove the original statement.

The intersection of $G(\mathbb{Z}_\ell)$ with the $G(\mathbb{Q}_\ell)$-orbit of γ_0 is a finite union of $G(\mathbb{Z}_\ell)$-orbits, since it is compact (recall that γ_0 is regular semisimple) and the $G(\mathbb{Z}_\ell)$-orbits are open in this intersection; let $g_1, \ldots, g_s$ be representatives of these orbits, and let $A_i \in G(\mathbb{Q}_\ell)$ be elements satisfying $A_i g_i A_i^{-1} = \gamma_0$, so that $A_i g_i = \gamma_0 A_i$. We clear denominators; for each i, let $X_i \in M(\mathbb{Z}_\ell)$ be a scalar multiple of A_i. Then $X_i g_i = \gamma_0 X_i$, and we set

$$e(\gamma_0) = \max_{i \in \{1, \ldots, s\}} \{|\mathrm{ord}(\det X_i)|\}.$$

Now, suppose $n > 2d(\gamma_0)$, where $d(\gamma_0)$ is the valuation of the discriminant of γ_0, $e \geq e(\gamma_0)$, and $n \gg e$. We want to prove that with these assumptions, an element $\gamma \in G(\mathbb{Z}_\ell/\ell^n)$ satisfies

$$\gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_e} \pi_n(\gamma_0)$$

if and only if there exists a lift $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$.

One direction is easy: suppose there exists $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\widetilde{\gamma} \bmod \ell^n = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$. Then there exists $i \in \{1, \ldots, s\}$ such that $\widetilde{\gamma} \sim_{G(\mathbb{Z}_\ell)} g_i$. Therefore there exists $Y \in G(\mathbb{Z}_\ell)$ such that $Y\widetilde{\gamma} = g_i Y$.

Recall that, as above, there exists $X_i \in M(\mathbb{Z}_\ell)$ such that $X_i g_i = \gamma_0 X_i$. Then $Z := \pi_n^M(X_i Y)$ lies in $M(\mathbb{Z}_\ell/\ell^n)_e$ and satisfies the condition $Z\gamma = \pi_n(\gamma_0)Z$.

The other direction is a special case of Hensel's lemma. Since Hensel's lemma in this generality, though well-known, is surprisingly hard to find in the literature, we provide a detailed explanation with references.

For each n, let

$$R_{\gamma_0}(\mathbb{Z}_\ell/\ell^n) = \left\{ (A, \gamma) : A \in M(\mathbb{Z}_\ell/\ell^n), \gamma \in G(\mathbb{Z}_\ell/\ell^n), A\gamma = \pi_n^G(\gamma_0)A \right\} \subset M(\mathbb{Z}_\ell/\ell^n) \times G(\mathbb{Z}_\ell/\ell^n),$$

where π_n^G is the projection from Section 2D. This is a system of $(2g)^2$ equations in $8g^2$ variables (namely, the matrix entries of A and γ). Now Hensel's lemma as stated in [Bourbaki 1985, III.4.5., Corollaire 3, p. 271] applies directly, as follows. Let $n(\gamma_0)$ be the valuation of the minor formed by the first $(2g)^2$ columns of the Jacobian matrix of this system of equations at γ_0. By Hensel's lemma, if $n > 2n(\gamma_0)$ and $(A, \gamma) \in R_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)$, then there exists some $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{M(\mathbb{Z}_\ell)} \gamma_0$.

Since the core argument simply relies on the solvability, via Hensel's lemma, of a system of equations over $\mathbb{Z}_\ell$, it is also valid if $G(\mathbb{Z}_\ell)$ is replaced by $M(\mathbb{Z}_\ell)$. $\qquad \square$

Remark 3.3. We observe (though we do not need this observation in this paper) that $n(\gamma_0)$ in fact equals the valuation of the discriminant of γ_0, e.g., by the argument provided in [Kottwitz 2005, §7.2].

For $\gamma_0 \in G(\mathbb{Z}_\ell)$, let

$$C_{(d,n)}(\gamma_0) = \{\gamma \in G(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\}. \tag{3-1}$$

If $d = 0$, this coincides with the usual conjugacy class of $\pi_n(\gamma_0)$. As in Section 2D, let $\widetilde{C}_{(d,n)}(\gamma_0) = (\pi_n^G)^{-1}(C_{(d,n)}(\gamma_0))$ be the set of lifts of elements of $C_{(d,n)}(\gamma_0)$ to $G(\mathbb{Z}_\ell)$.

We also extend this notation to elements $\gamma_0 \in M(\mathbb{Z}_\ell)$:

$$C_{(d,n)}(\gamma_0) = \{\gamma \in M(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\}.$$

(If $\gamma_0 \in G(\mathbb{Z}_\ell) \subset M(\mathbb{Z}_\ell)$, the two notions coincide and thus there is no ambiguity.)

Corollary 3.4. (a) *Suppose $\gamma_0 \in G(\mathbb{Z}_\ell)$. There exists $d = d(\gamma_0)$ such that, if $n \gg 0$, then*

$$C_{(d,n)}(\gamma_0) = \pi_n(G(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0).$$

Moreover,

$$G(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0 = \bigcap_{n \geq 0} \widetilde{C}_{(d,n)}(\gamma_0).$$

(b) *Suppose $\gamma_0 \in M(\mathbb{Z}_\ell)$. There exists $d = d(\gamma_0)$ such that, if $n \gg 0$, then*

$$C_{(d,n)}(\gamma_0) = \pi_n(M(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0).$$

Moreover,

$$M(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0 = \bigcap_{n \geq 0} \widetilde{C}_{(d,n)}(\gamma_0).$$

Proof. This is a direct consequence of Lemma 3.2. $\qquad \square$

3B. *Stable (twisted) conjugacy.* In this section, we further assume that $[X, \lambda]$ is a principally polarized abelian variety with commutative endomorphism ring for which $\frac{1}{2}$ is not a slope of the Newton polygon of X. (Again, any ordinary simple principally polarized abelian variety satisfies these hypotheses.)

Recall the definition of K and K^+, as well as the discussion of T, from Section 2B. By a prime of K (or K^+) lying over p we mean a prime $\mathfrak{p}$ of some K_i (respectively, K_i^+) lying over p, and we write $K_\mathfrak{p}$ for $K_{i,\mathfrak{p}}$. With this convention, we then have $K \otimes \mathbb{Q}_p \cong \bigoplus K_\mathfrak{p}$.

Lemma 3.5. *Let $\mathfrak{p}^+$ be a prime of K^+ lying over p. Then $\mathfrak{p}^+$ splits in K.*

Proof. This is standard. We work in the category of p-divisible groups up to isogeny. Then $X[p^\infty]$ has height $2g$, and comes equipped with an action by $K \otimes_\mathbb{Q} \mathbb{Q}_p$.

Corresponding to the decomposition $K^+ \otimes_\mathbb{Q} \mathbb{Q}_p \cong \bigoplus_{\mathfrak{p}^+ | p} K_{\mathfrak{p}^+}^+$ we have the decomposition

$$X[p^\infty] = \bigoplus X[\mathfrak{p}^{+\infty}].$$

Moreover, $X[\mathfrak{p}^{+\infty}]$ is a p-divisible group of height $2[K_{\mathfrak{p}^+}^+ : \mathbb{Q}_p]$, and self-dual (because $\mathfrak{p}^+ \mathcal{O}_K$ is stable under the Rosati involution). We now fix one $\mathfrak{p}^+$, and show that it must split in K.

Since $K_{\mathfrak{p}^+}^+$ is a field (and not just a $\mathbb{Q}_p$-algebra) of dimension $\frac{1}{2} \operatorname{ht}(X[\mathfrak{p}^{+\infty}])$, $X[\mathfrak{p}^{+\infty}]$ has at most two slopes. Since by hypothesis $\frac{1}{2}$ is not a slope of X, $X[\mathfrak{p}^{+\infty}]$ has exactly two slopes, say $\lambda = a/b$ and $1 - a/b$, where $\gcd(a, b) = 1$. Let m be the multiplicity of λ as a slope of $X[\mathfrak{p}^{+\infty}]$; then $mb = [K_{\mathfrak{p}^+}^+ : \mathbb{Q}_p]$. The endomorphism algebra of $X[\mathfrak{p}^{+\infty}]$ (again, in the category of p-divisible groups up to isogeny) is isomorphic to

$$\operatorname{End}(X[\mathfrak{p}^{+\infty}])^0 \cong \operatorname{Mat}_m(D_\lambda) \oplus \operatorname{Mat}_m(D_{1-\lambda}),$$

where D_λ is the central simple $\mathbb{Q}_p$-algebra with Brauer invariant λ. In particular, any subfield L of $\operatorname{End}(X[\mathfrak{p}^{+\infty}])^0$ satisfies $[L : \mathbb{Q}_p] \leq mb = [K_{\mathfrak{p}^+}^+ : \mathbb{Q}_p]$. Since $K \otimes_{K^+} K_{\mathfrak{p}^+}^+$ acts on $X[\mathfrak{p}^{+\infty}]$, we must have $K \otimes_{K^+} K_{\mathfrak{p}^+}^+ \cong K_{\mathfrak{p}^+}^+ \oplus K_{\mathfrak{p}^+}^+$, as claimed. $\square$

Corollary 3.6.
$$T_{\mathbb{Q}_p}^{\operatorname{der}} \cong \bigoplus_{\mathfrak{p}^+} \boldsymbol{R}_{K_{\mathfrak{p}^+}^+/\mathbb{Q}_p} \mathbb{G}_m.$$

Proof. Since $T^{\operatorname{der}} = \boldsymbol{R}_{K^+/\mathbb{Q}} \boldsymbol{R}_{K/K^+}^{(1)} \mathbb{G}_m$, using Lemma 3.5 we find

$$T_{\mathbb{Q}_p}^{\operatorname{der}} = \boldsymbol{R}_{K^+ \otimes \mathbb{Q}_p/\mathbb{Q}_p} \boldsymbol{R}_{K \otimes \mathbb{Q}_p/K^+ \otimes \mathbb{Q}_p}^{(1)} \mathbb{G}_m \cong \bigoplus_{\mathfrak{p}^+ | p} \boldsymbol{R}_{K_{\mathfrak{p}^+}^+/\mathbb{Q}_p} \boldsymbol{R}_{K \otimes_{K^+} K_{\mathfrak{p}^+}^+/K_{\mathfrak{p}^+}^+}^{(1)} \mathbb{G}_m.$$

If L is any field then $\boldsymbol{R}_{L \oplus L/L} \mathbb{G}_m \cong \mathbb{G}_{m,L} \oplus \mathbb{G}_{m,L}$; the norm map $\boldsymbol{R}_{L \oplus L/L} \mathbb{G}_m \to \mathbb{G}_m$ is given by multiplication of components; and so $\boldsymbol{R}_{L \oplus L/L}^{(1)} \mathbb{G}_m$ is isomorphic to $\mathbb{G}_{m,L}$, where the latter is embedded in the former via (id, inv). $\square$

Recall that the twisted conjugacy class of $\operatorname{N}(\delta_0)$ is defined over $\mathbb{Q}_p$, and consequently contains a $G(\mathbb{Q}_p)$-rational representative [Kottwitz 1982, p. 799, Theorem 4.4]; and we have adjusted δ_0 in its twisted conjugacy class to ensure that $\gamma_0 := \operatorname{N}(\delta_0) \in G(\mathbb{Q}_p)$ (2-2).

Lemma 3.7. *The stable conjugacy class of γ_0 consists of a single conjugacy class, and the stable σ-conjugacy class of δ_0 consists of a single σ-conjugacy class.*

Proof. To prove the first claim, it suffices (by [Kottwitz 1982, p. 788]) to show that $H^1(\mathbb{Q}_p, T)$ vanishes. By taking the long exact sequence of cohomology of the top row of (2-3), and then invoking Hilbert 90 and Corollary 3.6, we find that $H^1(\mathbb{Q}_p, T)$ does in fact vanish.

For the second claim, it similarly suffices to show that the first cohomology of the twisted centralizer $G_{\delta_0,\sigma}$ vanishes [Kottwitz 1982, p. 805]. However, the twisted centralizer of an element is always an inner form of the (usual) centralizer of its norm [Kottwitz 1982, Lemma 5.8]. In our case, the centralizer $T = G_{\gamma_0}$ is a torus, and thus admits no nontrivial inner forms. We conclude again that $H^1(\mathbb{Q}_p, G_{\delta_0,\sigma})$ is trivial. $\square$

4. Ratios

4A. *Definitions.* For $\ell \neq p$, we define a local ratio $\nu_\ell([X, \lambda])$ designed to measure the extent to which the conjugacy class of $\gamma_{X_0/\mathbb{F}_q}$, as an element of $G(\mathbb{Z}_\ell/\ell)$, is more or less prevalent than a randomly chosen conjugacy class. More precisely, to measure this probability, we consider the finite group $G(\mathbb{Z}_\ell/\ell^n)$ for sufficiently large n, and recall that our notion of "conjugacy" in this group is not the usual conjugacy but the relation $\sim_{M(\mathbb{Z}_\ell/\ell^n)_e}$ defined above in Section 2D. For $\ell = p$, the element $\gamma_{X_0/\mathbb{F}_q}$ is not in $G(\mathbb{Z}_p)$, and we use $M(\mathbb{Z}_p)$ instead; but this has no effect on the definition since our notion of "conjugacy" in $G(\mathbb{Z}_p/p^n)$ already uses $M(\mathbb{Z}_p)$.

Recall the definition of $C_{(d,n)}(\gamma_0)$ from (3-1), and that $C_n(\gamma_0) := C_{(0,n)}(\gamma_0)$ is the actual conjugacy class of $\pi_n(\gamma_0)$ in $G(\mathbb{Z}_\ell/\ell^n)$.

Definition 4.1. For each finite place ℓ, including $\ell = p$, using the shorthand $\gamma_0 := \gamma_{X/\mathbb{F}_q,\ell} \in M(\mathbb{Z}_\ell)$, set

$$\nu_\ell([X, \lambda]) = \nu_\ell([X, \lambda], \mathbb{F}_q) = \lim_{d \to \infty} \lim_{n \to \infty} \frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}. \tag{4-1}$$

At infinity, define

$$\nu_\infty([X, \lambda]) = \nu_\infty([X, \lambda], \mathbb{F}_q) = \frac{|D(\gamma_0)|_\infty^{\frac{1}{2}}}{(2\pi)^g}, \tag{4-2}$$

where $|\cdot|_\infty$ is the usual real absolute value.

Remark 4.2. So far we have avoided using the fact that there exists a rational element $\gamma_0 \in G(\mathbb{Q})$ as in Remark 2.2, and treated γ_0 as an element of $G(\mathbb{A}_f)$. We can continue doing so, and then for (4-2) simply *define* the archimedean absolute value of its discriminant by $|D(\gamma_0)|_\infty := \prod_\ell |D(\gamma_{X/\mathbb{F}_q,\ell})|_\ell^{-1}$.

The ratios stabilize for large enough d and n, and thus the limits are not, strictly speaking, necessary. In fact, for $\ell \neq 2, p$ and not dividing the discriminant of γ_0, the ratios stabilize right away, at $d = 0$ and $n = 1$, as the next two lemmas show.

Lemma 4.3. *If $\ell \nmid D(\gamma_0)$ and $n \gg 0$, then $C_{(d,n)}(\gamma_0) = C_n(\gamma_0)$.*

Proof. Recall that all our notation assumes that ℓ is fixed. Clearly $C_n(\gamma_0) \subseteq C_{(d,n)}(\gamma_0)$. Conversely, suppose n is sufficiently large as in Lemma 3.2 and that $\gamma \in C_{(d,n)}(\gamma_0)$. Then there exists some $\widetilde{\gamma} \in \pi_n^{-1}(\gamma)$ such that $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$. By Lemma 3.1, $\widetilde{\gamma} \sim_{G(\mathbb{Z}_\ell)} \gamma_0$, and so $\gamma \sim_{G(\mathbb{Z}_\ell/\ell^n)} \gamma_0$. $\qquad\square$

Lemma 4.4. *If $\ell \nmid D(\gamma_0)$, $\ell \neq 2$, and $d \gg_{\gamma_0} 0$, then for $n \geq 1$,*

$$\frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#\{\gamma \in G(\mathbb{Z}_\ell/\ell) : \gamma \sim \gamma_0\}}{\#G(\mathbb{Z}_\ell/\ell)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell)}. \tag{4-3}$$

Proof. By Lemma 4.3, the left-hand side of (4-3) is

$$\frac{\#G(\mathbb{Z}_\ell/\ell^n)/\#G_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}{\#G_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)}.$$

Since $\pi_1(\gamma_0)$ is regular, the centralizer G_{γ_0} is smooth over $\mathbb{Z}_\ell$ of relative dimension $g+1$. Since the same is true of the scheme $\mathbb{A}_G/\mathbb{Z}_\ell$, the result now follows. $\qquad\square$

4B. *From ratios to integrals.* Fix a prime ℓ (possibly $\ell = p$ or $\ell = 2$). (In this subsection, as above, all quantities depend on this notationally suppressed prime.) Recall (2-4) the canonical map $\mathfrak{c} : G \to A$ from G to its Steinberg quotient. The fibres of this map over regular points are stable orbits of regular semisimple elements. Define a system of neighbourhoods of $\mathfrak{c}(\gamma_0)$ inside $\mathbb{A}_G(\mathbb{Z}_\ell)$ by

$$\widetilde{U}_n(\gamma_0) = \widetilde{\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))} = (\pi_n^{\mathbb{A}_G})^{-1}\big(\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))\big).$$

In other words,

$$\widetilde{U}_n(\gamma_0) = \{a = (a_1, \ldots, a_g, \eta) \in \mathbb{A}_G(\mathbb{Z}_\ell) \mid a_i \equiv a_i(\gamma_0) \bmod \ell^n, \, \eta \equiv \eta(\gamma_0) \bmod \ell^n\}.$$

These definitions and (3-1) are summarized by the diagram

$$
\begin{array}{ccc}
\widetilde{C}_{(d,n)}(\gamma_0) \subset M(\mathbb{Z}_\ell) & \xrightarrow{\;\;\mathfrak{c}\;\;} & \mathbb{A}_G(\mathbb{Z}_\ell) \supset \widetilde{U}_n(\gamma_0) \\
\downarrow{\scriptstyle \pi_n^M} & & \downarrow{\scriptstyle \pi_n^{\mathbb{A}_G}} \\
C_{(d,n)}(\gamma_0) \subset M(\mathbb{Z}_\ell/\ell^n) & \xrightarrow{\;\;\mathfrak{c}_n\;\;} & \mathbb{A}_G(\mathbb{Z}/\ell^n) \ni \pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))
\end{array}
\tag{4-4}
$$

where $\mathfrak{c}_n : G(\mathbb{Z}/\ell^n) \to \mathbb{A}_G(\mathbb{Z}/\ell^n)$ is the map sending an element to the coefficients of its characteristic polynomial mod ℓ^n. The diagram of maps commutes since reduction mod ℓ^n is a ring homomorphism, and the map $\mathfrak{c}$ is polynomial in the matrix entries of γ. (The diagram of subsets need not commute, though.) We also note that when $\ell \neq p$, the sets $\widetilde{C}_{(d,n)}(\gamma_0)$ and $C_{(d,n)}(\gamma_0)$ are contained in $G(\mathbb{Z}_\ell)$ and $G(\mathbb{Z}_\ell/\ell^n)$, respectively, since $\mathrm{ord}_\ell(\det(\gamma_0)) = 0$ in this case, and this is also true for all elements that are congruent to any conjugate of γ_0.

By definition of the geometric measure, for any open subset $B \subset G(\mathbb{Z}_\ell)$ we have

$$\mathrm{vol}_{\mu^{\mathrm{geom}}}\big(B \cap \mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))\big) = \lim_{n \to \infty} \frac{\mathrm{vol}_{|d\omega_G|}\big(\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap B\big)}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}. \tag{4-5}$$

Recall that each stable orbit $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$ is a finite disjoint union of rational orbits. Each rational orbit being an open subset of the stable orbit, we may and do define geometric measure on each rational orbit, by restriction.

In simple terms, the sets $\widetilde{U}_n$ form a system of neighbourhoods of the point $\mathfrak{c}(\gamma_0) \in \mathbb{A}_G$; the set $\widetilde{C}_{(d,n)}(\gamma_0)$ can be thought of as the intersection of a neighbourhood of the orbit of γ_0 with $G(\mathbb{Z}_\ell)$; the set $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ is the stable orbit of γ_0. The following lemma gives the precise relationships between all these sets.

Lemma 4.5. (a) *Let $\ell \neq p$. For large enough d and n (depending on γ_0), we have*

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell) = \bigcup_{\gamma' \sim_{G(\bar{\mathbb{Q}}_\ell)} \gamma_0} \widetilde{C}_{d,n}(\gamma'), \tag{4-6}$$

where γ' runs over a set of representatives of $G(\mathbb{Q}_\ell)$-conjugacy classes in the stable conjugacy class of γ_0 whose $\mathbb{Q}_\ell$-orbits intersect $G(\mathbb{Z}_\ell)$, so that we may take the elements γ' to lie in $G(\mathbb{Z}_\ell)$.

(b) *When n is sufficiently large (depending on γ_0), the sets $\widetilde{C}_{d,n}(\gamma')$ above are disjoint.*

(c) *Let μ_G^{SO} be the Serre–Oesterlé measure on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$, viewed as a submanifold of $M(\mathbb{Z}_p)$. Then $\mathrm{vol}_{\mu_G^{\mathrm{SO}}}(\widetilde{C}_{d,n}(\gamma_0)) = \ell^{-n\dim(G)}\#C_{d,n}(\gamma_0)$; in particular, $\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0)) = \ell^{-n\dim(G)}\#C_{d,n}(\gamma_0)$ if $\ell \neq p$.*

Proof. (a) This is an easy consequence of the fact that two regular semisimple elements of $G(\mathbb{Q}_\ell)$ are stably conjugate if and only if their characteristic polynomials coincide. In our notation,

$$\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0)) = \bigsqcup_{\gamma' \sim_{G(\bar{\mathbb{Q}}_\ell)} \gamma_0} (^{G(\mathbb{Q}_\ell)}\gamma'),$$

where $^{G(\mathbb{Q}_\ell)}\gamma'$ denotes the rational conjugacy class of γ' in $G(\mathbb{Q}_\ell)$ as before. Now, we will describe both the left-hand side and the right-hand side of (4-6) as the set of elements $\gamma \in G(\mathbb{Z}_\ell)$ whose characteristic polynomial is congruent to that of γ_0 mod ℓ^n. Indeed, on the left-hand side, by definition, $\gamma \in \mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell)$ if and only if $\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma)) = \pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))$. By the commutativity of (4-4), this is equivalent to $\mathfrak{c}_n(\pi_n^G(\gamma)) = \mathfrak{c}_n(\pi_n^G(\gamma_0))$, i.e., the characteristic polynomials of γ and γ_0 are congruent mod ℓ^n. On the right-hand side, given $\gamma' \in G(\mathbb{Z}_\ell)$, by Lemma 3.2, for d and n large enough,[1] we have that $\gamma \in \widetilde{C}_{(d,n)}(\gamma')$ if and only if there exists $\gamma'' \in G(\mathbb{Z}_\ell)$ such that $\gamma'' \equiv \gamma' \mod \ell^n$ and γ'' is $G(\mathbb{Q}_\ell)$-conjugate to γ. Taking the union of these sets as γ' runs over the set of integral representatives of $G(\mathbb{Q}_\ell)$-conjugacy classes in the stable class of γ_0, we obtain the set of all elements $\gamma \in G(\mathbb{Z}_\ell)$ that are congruent modulo ℓ^n to an element of $G(\mathbb{Z}_\ell)$ that is stably conjugate to γ_0, i.e., to an element having the same characteristic polynomial as γ_0. This means that $\mathfrak{c}_n(\pi_n^G(\gamma)) = \mathfrak{c}_n(\pi_n^G(\gamma_0))$, completing the proof of the first statement.

(b) Since the orbits of regular semisimple elements are closed in the ℓ-adic topology, distinct orbits have disjoint neighbourhoods.

---

[1]*Large enough* depends on γ', but only through its discriminant. Since stably conjugate elements have the same discriminant, ultimately this only depends on γ_0.

(c) The map $\pi_n^M : \widetilde{C}_{(d,n)}(\gamma_0) \to C_{(d,n)}(\gamma_0)$ is surjective, so $\widetilde{C}_{(d,n)}(\gamma_0)$ can be thought of as a disjoint union of fibres of π_n^M. Since M is a smooth scheme over $\mathbb{Z}_\ell$, each fibre of π_n^M has volume $\ell^{-n \dim(G)}$ with respect to the measure μ^{SO} (see [Serre 1981]). The first statement follows. Moreover, as discussed above in Section 2E, on $G(\mathbb{Z}_\ell)$, the measures μ^{SO} and $\mu_{|\omega_G|}$ coincide. For $\ell \neq p$, we have $\widetilde{C}_{(d,n)}(\gamma_0) \subset G(\mathbb{Z}_\ell)$, which completes the proof. $\square$

Recall that ϕ_0 is the characteristic function of $G(\mathbb{Z}_\ell)$.

Corollary 4.6. *Let $\ell \neq p$. Then there exists $d(\gamma_0)$ such that for $d \geq d(\gamma_0)$*

$$O_{\gamma_0}^{\mathrm{geom}}(\phi_0) = \lim_{n \to \infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

Proof. The orbital integral, by definition, calculates the volume of the set of integral points in the rational orbit of γ_0, with respect to the geometric measure on the orbit. Using Lemma 4.5(a)–(b), we write $\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell) = \bigsqcup_{\gamma'} \widetilde{C}_{d,n}(\gamma')$, where γ' are as in that lemma, with γ_0 being one of the elements γ'. The union on the right-hand side of (4-6) is a disjoint union of neighbourhoods of the individual orbits, intersected with $G(\mathbb{Z}_\ell)$. The statement follows from the equality (4-5), applied to the set $B := \widetilde{C}_{d,n}(\gamma_0)$. $\square$

Corollary 4.7. *For $\ell \neq p$, the Gekeler ratio (4-1) is related to the geometric orbital integral by*

$$\nu_\ell([X, \lambda]) = \frac{\ell^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} O_{\gamma_0}^{\mathrm{geom}}(\phi_0).$$

Proof. Note that at a finite level n (and for d large enough so that the equalities in all the previous lemmas hold), the denominator in (4-1) is

$$\frac{\#G(\mathbb{Z}_\ell/\ell^n)}{\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell^n)\#\mathbb{G}_m(\mathbb{Z}_\ell/\ell^n)}{\ell^{(\mathrm{rank}(G)-1)n}\#\mathbb{G}_m(\mathbb{Z}_\ell/\ell^n)} = \frac{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell^n)}{\ell^{(\mathrm{rank}(G)-1)n}} = \frac{\ell^{(\dim(G)-1)(n-1)}\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)}{\ell^{(\mathrm{rank}(G)-1)n}}.$$

By Lemma 4.5(c), we have $\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0)) = \#C_{d,n}(\gamma_0)/\ell^{n \dim(G)}$, and by definition of the measure on the Steinberg quotient, $\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0)) = \ell^{-n \, \mathrm{rank}(G)}$ (here we are using the fact that $|\eta(\gamma_{X,\ell})| = 1$ for $\ell \neq p$, so the absolute value of the $\mathbb{G}_m$-coordinate is 1 on $\widetilde{U}_n$).

Then for a given level n, we have

$$\begin{aligned}
\frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} &= \frac{\ell^{n \dim(G)} \, \mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))\ell^{(\mathrm{rank}(G)-1)n}}{\ell^{(\dim(G)-1)(n-1)}\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} \\
&= \frac{\ell^{\dim(G)-1}}{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.
\end{aligned}$$

The result now follows from Corollary 4.6. $\square$

4C. *Calculation at p.* Recall that we have fixed a maximal split torus $T_{\mathrm{spl}} \subset G$. For any cocharacter $\lambda \in X_*(T_{\mathrm{spl}})$ (and any power $q = p^e$ of p), let $\psi_\lambda = \psi_{\lambda,q}$ be the characteristic function of the double coset

$$D_{\lambda,q} = G(\mathbb{Z}_q)\lambda(p)G(\mathbb{Z}_q).$$

By the Cartan decomposition, the collection of all ψ_λ is a basis for $\mathcal{H}_G = \mathcal{H}_{G,\mathbb{Q}_q}$, the Hecke algebra of functions on $G(\mathbb{Q}_q)$ which are bi-$G(\mathbb{Z}_q)$-invariant.

Let μ_0 be the cocharacter $p \mapsto \mathrm{diag}(p, \ldots, p, 1, \ldots, 1)$; it is the cocharacter associated to the Shimura variety $\mathcal{A}_g$. Define

$$\psi_{q,p} = \psi_{\mu_0,q} = \mathbb{1}_{G(\mathbb{Z}_q)\,\mathrm{diag}(p,\ldots,p,1,\ldots,1)G(\mathbb{Z}_q)} \quad \text{and} \quad \phi_{q,p} = \psi_{e\mu_0,p} = \mathbb{1}_{G(\mathbb{Z}_p)\,\mathrm{diag}(q,\ldots,q,1,\ldots,1)G(\mathbb{Z}_p)}.$$

Recall that $\delta_0 = \delta_{X/\mathbb{F}_q}$ represents the absolute Frobenius of X, and that $\gamma_0 := \mathrm{N}\,\delta_0$ lies in $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$ (2-2).

Lemma 4.8. *Let $[X, \lambda]/\mathbb{F}_q$ be a principally polarized abelian variety. Suppose that either X is ordinary or that $q = p$ (and thus $e = 1$). Then*

$$^{G(\mathbb{Q}_p)}\gamma_0 \cap M(\mathbb{Z}_p) \subseteq D_{e\mu_0,p}.$$

Proof. We identify $X_*(T_{\mathrm{spl}})$ with

$$\{\underline{a} = (a_1, \ldots, a_{2g}) \in \mathbb{Z}^{2g} : a_i + a_{g+i} = a_j + a_{g+j} \text{ for } 1 \le i, j \le g\}. \tag{4-7}$$

Suppose $\gamma \in D_{\underline{a},p} \subseteq G(\mathbb{Q}_p)$. Then $\mathrm{ord}_p(\eta(\gamma))$ is the common value of $a_i + a_{g+i}$; and γ stabilizes $V \otimes \mathbb{Z}_p$ — that is, $\gamma \in M(\mathbb{Z}_p)$ — if and only if each a_i satisfies $a_i \ge 0$.

Let $f(\underline{a}) = \#\{i : a_i = 0\}$. If $\alpha \in D_{\underline{a}} \cap M(\mathbb{Z}_p)$, then $f(\underline{a})$ is the rank of $\pi_1(\alpha)$ as an endomorphism of V/pV.

With these preparations, suppose $\gamma \in {}^{G(\mathbb{Q}_p)}\gamma_0 \cap M(\mathbb{Z}_p)$. Note that we have $a_i + a_{g+i} = e$.

First, suppose X is ordinary. Then exactly g eigenvalues of γ are p-adic units. Consequently, if $\gamma \in D_{\underline{a}}$, then $f(\underline{a}) = g$. The only $\underline{a}$ as in (4-7) compatible with the symmetry and integrality constraints is $(e, \ldots, e, 0, \ldots, 0)$.

Second, suppose X has arbitrary Newton polygon but that $e = 1$. Again, the only $\underline{a}$ such that $a_i + a_{g+i} = e = 1$ and each a_i is nonnegative is $(1, \ldots, 1, 0, \ldots, 0)$. $\qquad\square$

Lemma 4.9. *Suppose that $[X, \lambda]/\mathbb{F}_q$ is an ordinary, simple, principally polarized abelian variety. Then*

$$T O_{\delta_0}(\psi_{q,p}) = O_{\gamma_0}(\phi_{q,p}).$$

Proof. There is a base change map $b = b_{G,\mathbb{Q}_q/\mathbb{Q}_p} : \mathcal{H}_{G,\mathbb{Q}_q} \to \mathcal{H}_{G,\mathbb{Q}_p}$. The fundamental lemma asserts that if $\psi \in \mathcal{H}_{G,\mathbb{Q}_q}$, then stable twisted orbital integrals for ψ match with stable orbital integrals for $b\psi$. For our δ_0 and γ_0, the adjective *stable* is redundant (Lemma 3.7), the case of the fundamental lemma we need is [Clozel 1990, Theorem 1.1], and we have

$$T O_{\delta_0}(\psi_{q,p}) = O_{\gamma_0}(b\psi_{q,p}). \tag{4-8}$$

While we will stop short of computing $b\psi_{q,p}$, we will find a function that agrees with it on the orbit $^{G(\mathbb{Q}_p)}\gamma_0$.

The Satake transformation is an algebra homomorphism $\mathfrak{S} : \mathcal{H}_{G,\mathbb{Q}_q} \to \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_q}$ which maps $\mathcal{H}_{G,\mathbb{Q}_q}$ isomorphically onto the subring $\mathcal{H}^W_{T_{\mathrm{spl}},\mathbb{Q}_q}$ of invariants under the Weyl group. It is compatible with base

change, in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}_{G,\mathbb{Q}_q} & \xrightarrow{\ \mathfrak{S}\ } & \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_q} \cong \mathbb{C}[X_*(T_{\mathrm{spl}})] \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
\mathcal{H}_{G,\mathbb{Q}_p} & \xrightarrow{\ \mathfrak{S}\ } & \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_p}
\end{array}
$$

We exploit the following data about the Satake transform and the base change map.

Under the canonical identification of $X_*(T_{\mathrm{spl}})$ and $X^*(\widehat{T}_{\mathrm{spl}})$, the character group of the dual torus, $\lambda \in X_*(T_{\mathrm{spl}})$ gives rise to a character of $\widehat{T}_{\mathrm{spl}}$, and thus a representation V_λ of $\widehat{G}$; let χ_λ be its trace. Then

$$
\mathfrak{S}(\psi_{\mu,q}) = q^{\langle \mu, \rho \rangle} \chi_\mu + \sum_{\lambda < \mu} a(\mu, \lambda) \chi_\lambda
$$

for certain numbers $a(\mu, \lambda)$, where as usual ρ is the half-sum of positive roots [Gross 1998, (3.9)].

On one hand, following Gross [1998, (3.15)] and Kottwitz [1984a, (2.1.3)], we observe that the weight $\mu_0 = (1, \ldots, 1, 0, \ldots, 0)$ is minuscule, and therefore

$$
\mathfrak{S}(\psi_{\mu_0,q}) = q^{\langle \mu_0, \rho \rangle} \chi_{\mu_0}.
$$

If we think of elements of $\mathbb{C}[X_*(T_{\mathrm{spl}})]^W$ as polynomials in $2g$ variables $z_1, \ldots, z_{2g}$, then (essentially by definition of the highest weight and the fact that the multiplicity of the highest weight in an irreducible representation is 1 — in our case the representation in question is in fact the oscillator representation [Gross 1998, (3.15)]), we find that the leading term of $\mathfrak{S}(\psi_{\mu_0,q})$ is $q^{\langle \mu_0, \rho \rangle} z_1 \cdots z_g$. By definition, the base change map takes $f \in \mathbb{C}[z_1, \ldots, z_{2g}, z_1^{-1}, \ldots, z_{2g}^{-1}]^W$ to $f(z_1^e, \ldots, z_{2g}^e)$. Then

$$
b(\mathfrak{S}(\psi_{q,p})) = q^{\langle \mu_0, \rho \rangle} z_1^e \cdots z_g^e + \sum_{\lambda < e\mu_0} a(e\mu_0, \lambda) \chi_\lambda.
$$

On the other hand, we have

$$
\mathfrak{S}(\phi_{q,p}) = p^{\langle e\mu_0, \rho \rangle} \chi_{e\mu_0} + \sum_{\lambda < e\mu_0} b(e\mu_0, \lambda) \chi_\lambda = q^{\langle \mu_0, \rho \rangle} z_1^e \cdots z_g^e + \sum_{\lambda < e\mu_0} c(e\mu_0, \lambda) \chi_\lambda.
$$

In these formulas, $a(e\mu_0, \lambda)$, $b(e\mu_0, \lambda)$ and $c(e\mu_0, \lambda)$ are coefficients of lower-weight monomials that are ultimately irrelevant to our calculation. In particular,

$$
\phi_{q,p} - \mathfrak{S}^{-1}\big(b(\mathfrak{S}(\psi_{q,p}))\big)
$$

vanishes on $D_{e\mu_0,p} = G(\mathbb{Z}_p) e\mu_0(p) G(\mathbb{Z}_p)$.

The last point to note is that the intersection of the support of this difference $\phi_{q,p} - \mathfrak{S}^{-1}\big(b(\mathfrak{S}(\psi_{q,p}))\big)$ with the orbit of γ_0 is contained in $M(\mathbb{Z}_p)$. Once we have shown this, the desired result follows from the fundamental lemma (4-8) combined with Lemma 4.8. We start by observing that since the multiplier is a multiplicative map, it is constant on double $G(\mathbb{Z}_p)$-cosets. Therefore, for any double coset $D_{\underline{a},p}$ such that $D_{\underline{a},p} \cap {}^{G(\mathbb{Q}_p)}\gamma_0 \neq \varnothing$, we have $a_i + a_{g+i} = e$. Now suppose $\lambda \leftrightarrow \underline{a}$ is a dominant weight satisfying this condition and further satisfying $\lambda \leq e\mu_0$. Then we have $a_1 \geq a_2 \geq \cdots \geq a_g$ and $a_g \geq 0$ because λ

is dominant; and on the other hand, $e - a_1 \geq e - a_2 \geq \cdots \geq e - a_g$, and $e - a_g \geq 0$ because of the condition $\lambda \leq e\mu_0$. Therefore, in particular, $a_{g+1}, \ldots, a_{2g}$ are nonnegative, and thus $D_{\lambda,p} \subset M(\mathbb{Z}_p)$ (and in fact, we have also shown that $a_1 = \cdots = a_g$). $\qquad\square$

Lemma 4.10. *Suppose that either X is ordinary or that $q = p$. Then there exists $d(\gamma_0)$ such that*

$$O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}) = \lim_{n \to \infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d(\gamma_0),n}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

Proof. Suppose that X is ordinary (but q is an arbitrary power of p). By Lemma 3.7, $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ is a single $G(\mathbb{Q}_p)$-conjugacy class; the same argument shows this is true for elements in a small neighbourhood of $\mathfrak{c}(\gamma_0)$. So, using (4-5), $O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p})$ equals $\lim_{n\to\infty}\big(\mathrm{vol}_{|d\omega_G|}(\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0))\cap D_{e\mu_0,p})/\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))\big)$. By Lemma 4.8, we have

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap D_{e\mu_0,p} = \mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap M(\mathbb{Z}_p).$$

Therefore, all we need to show is that for large enough d and n, we have

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap M(\mathbb{Z}_p) = \widetilde{C}_{d,n}(\gamma_0); \tag{4-9}$$

but this is essentially Corollary 3.4(b).

The case where $q = p$ follows from Corollary 4.6 and the second case of Lemma 4.8. $\qquad\square$

Lemma 4.11. *On the double coset $D_{e\mu_0,p}$ we have*

$$|d\omega_G| = q^{\frac{1}{2}g(g+1)+1}\mu^{\mathrm{SO}}. \tag{4-10}$$

Proof. Let $K = G(\mathbb{Z}_p)$. First, observe that the measure μ^{SO} on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$ is both left- and right-K-invariant (since multiplication by an element of $G(\mathbb{Z}_p)$ yields a bijection on mod p^n-points). Consider the decomposition of $D_{e\mu_0,p}$ into, say, left K-cosets: $D_{e\mu_0,p} = \bigsqcup_{i=1}^s g_i K$ (the number s of these cosets was computed by Iwahori and Matsumoto but is not needed here). It follows from left K-invariance of μ^{SO} that $\mu^{\mathrm{SO}}(g_i K)$ is the same for all i.

Second, the measure $|d\omega_G|$ is normalized so that each K-coset has volume $\#G(\mathbb{F}_p)$. Thus, in order to compare the measures μ^{SO} and $|d\omega_G|$, we need to compare the cardinality $\#\pi_n(g_i K)$ of the reduction mod p^n of any such coset $g_i K$ that is contained in $D_{e\mu_0,p}$ with $\#G(\mathbb{F}_p)$, for sufficiently large n. (Note that $n = 1$ is insufficient, because for all such cosets the reduction mod p of any matrix in gK would be of lower rank. One needs to go to $n > e$ for the ratios $\#\pi_n(gK)/p^{n\dim(G)}$ to stabilize.) Since the answer does not depend on g_i, we can take $g_0 = e\mu_0(p) = \mathrm{diag}(q, \ldots, q, 1, \ldots, 1)$. In other words, we need to compute the cardinality of the fibre of the map

$$\varphi_q : G(\mathbb{Z}/p^n) \to M(\mathbb{Z}/p^n), \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} qA & qB \\ C & D \end{bmatrix}.$$

For simplicity, we would like to move the calculation to the Lie algebra. Let $n \gg e$. Observe that if $\varphi_q(\gamma_1) = \varphi_q(\gamma_2)$ for $\gamma_1, \gamma_2 \in G(\mathbb{Z}/p^n)$, then $\begin{bmatrix} qI_g & 0 \\ 0 & I_g \end{bmatrix}(\gamma_1\gamma_2^{-1} - I) = 0$, where I_g is the $g \times g$-identity

matrix, and I is the identity matrix in M_{2g}. This implies, in particular, that $\gamma_1 \gamma_2^{-1} \equiv I \mod p^{n-e}$. Then we can write the truncated exponential approximation: $\gamma_1 \gamma_2^{-1} = I + X + \frac{1}{2}X^2 + \cdots$ for some $X \in \mathfrak{g}(\mathbb{Z}_p)$; in particular, there exists $X \in \mathfrak{g}(\mathbb{Z}_p)$ such that $\gamma_1 \gamma_2^{-1} \equiv I + X \mod p^{2(n-e)}$, and thus the kernel of the map φ_q is in bijection with the set of $(X \mod p^n)$ for $X \in \mathfrak{g}(\mathbb{Z}_p)$ such that $\begin{bmatrix} qI_g & 0 \\ 0 & I_g \end{bmatrix} X \equiv 0 \mod p^n$.

We have $\mathfrak{g} = \mathfrak{sp}_{2g} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the 1-dimensional Lie algebra of the centre. It will be convenient to decompose it further: let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sp}_{2g}$ consisting of diagonal matrices, and let V consist of matrices whose diagonal entries are all zero; then

$$\mathfrak{g} = (\mathfrak{z} \oplus \mathfrak{h}) \oplus V.$$

Consider the action of multiplication by $\begin{bmatrix} qI_g & 0 \\ 0 & I_g \end{bmatrix}$ on each term of this direct sum decomposition.

On the term $\mathfrak{z} \oplus \mathfrak{h}$ it acts by $\mathrm{diag}(a_1, \ldots, a_{2g}) \mapsto \mathrm{diag}(qa_1, \ldots, qa_g, a_{g+1}, \ldots, a_{2g})$, which in the $\mathfrak{z} \oplus \mathfrak{h}$-coordinates can be written as (recalling that $a_i + a_{g+i} = z$ is independent of i):

$$\frac{z}{2} \oplus \left(\frac{z}{2} - a_{g+1}, \ldots, \frac{z}{2} - a_{2g}, -\frac{z}{2} + a_{g+1}, \ldots, -\frac{z}{2} + a_{2g} \right)$$

$$\mapsto \frac{qz}{2} \oplus \left(\frac{qz}{2} - \frac{(q+1)a_{g+1}}{2}, \ldots, \frac{qz}{2} - \frac{(q+1)a_{2g}}{2}, -\frac{qz}{2} + \frac{(q+1)a_{g+1}}{2}, \ldots, -\frac{qz}{2} + \frac{(q+1)a_{2g}}{2} \right).$$

The only points $(z, a_{g+1}, \ldots, a_{2g})$ that are killed (mod p^n) by this map are of the form $(z', 0, \ldots, 0)$ with $qz' = 0$; so there are q of them.

Next consider an element $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in V$. Then A is determined by D, and B is skew-symmetric (up to a permutation of rows and columns). Multiplication by $\begin{bmatrix} qI_g & 0 \\ 0 & I_g \end{bmatrix}$ scales each entry of A and B by a factor of q, and does not change C and D. Since A is determined by D, the elements X killed by this map are in bijection with symmetric matrices B with entries in $\mathbb{Z}/p^n$ that are killed by multiplication by q. Since the space of such matrices is a $\left(\frac{1}{2}g(g+1) \right)$-dimensional linear space, the number of such matrices B is $q^{\frac{1}{2}g(g+1)}$.

Thus, we have computed that $|d\omega_G| = q^{\frac{1}{2}g(g+1)+1} \mu^{\mathrm{SO}}$ on the double coset $D_{e\mu_0, p}$. Combining this with (4-11), we get

$$\nu_p([X, \lambda]) = \frac{qp^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} \frac{q^{-\frac{1}{2}g(g+1)-1} \, \mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))} = q^{-\frac{1}{2}g(g+1)} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}),$$

which completes the proof. $\qquad\square$

Corollary 4.12. *Suppose that either X is ordinary or that $q = p$. For $\ell = p$, the Gekeler ratio* (4-1) *is related to the geometric orbital integral by*

$$\nu_p([X, \lambda]) = q^{-\frac{1}{2}g(g+1)} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}).$$

Proof. First observe that $\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0)) = qp^{-n\,\mathrm{rank}(G)}$, since we are using the invariant measure on the $\mathbb{G}_m$-factor of $\mathbb{A}_G = \mathbb{A}^{\mathrm{rank}(G)-1} \times \mathbb{G}_m$, and for γ_0 (and therefore, for all points in $\widetilde{U}_n$), that coordinate is the

multiplier, with absolute value q^{-1}. Thus, by Lemma 4.5(c) and the same argument as in Corollary 4.7, we have that for $d > d(\gamma_0)$,

$$\nu_p([X, \lambda]) = \lim_{n \to \infty} \frac{\#C_{d,n}(\gamma_0)}{\#G(\mathbb{Z}/p^n\mathbb{Z})/\#\mathbb{A}_G(\mathbb{Z}/p^n\mathbb{Z})} = \frac{qp^{\dim(G)-1}}{\#G^{\mathrm{der}}(\mathbb{Z}/p\mathbb{Z})} \lim_{n \to \infty} \frac{\mathrm{vol}_{\mu^{\mathrm{SO}}}(\widetilde{C}_{d,n}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}. \quad (4\text{-}11)$$

The ratio inside the limit on the right-hand side is the same as the ratio in Lemma 4.10, except that the measure in the numerator is the Serre–Oesterlé measure μ^{SO} rather than the measure $|d\omega_G|$. (Both measures are defined on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$.) Thus, to prove the corollary, we just need to compute the conversion factor between the restrictions of the measures μ^{SO} and $|d\omega_G|$ to the support of $\phi_{q,p}$, which is the content of Lemma 4.11. $\square$

5. The product formula

Now that the relationship between the ratios ν_ℓ and orbital integrals (with respect to the geometric measure) is established, we can translate the formula of Langlands and Kottwitz (2-1) into a Siegel-style product formula for the ratios, thus obtaining our main theorem. Recall the notation of Section 2, in particular, the element $\gamma_{[X,\lambda]} \in G(\mathbb{A}_f)$ associated with the isogeny class of $[X, \lambda]$, and its centralizer $T = T_{[X,\lambda]}$. Here in order to ease the notation we drop all the subscripts $[X, \lambda]$. Note that there is some flexibility in the choice of the measures in the Langlands–Kottwitz formula, but the measures need to be normalized by normalizing the measures on $G(\mathbb{Q}_\ell)$ and on $T(\mathbb{Q}_\ell)$ separately. We will use the canonical measure $d\mu_G^{\mathrm{can}}$ on $G(\mathbb{Q}_\ell)$ for every prime ℓ, and the Tamagawa measure μ_T^{Tama} (defined in detail below) on $T(\mathbb{Q}_\ell)$ for all ℓ. This gives a convergent product measure globally, since the local orbital integrals equal 1 at almost all places with respect to this measure. Since Gekeler-style ratios are expressed in terms of the geometric measure on orbits, we need to calculate the conversion factor between the geometric measure and the quotient $\mu_G^{\mathrm{can}}/\mu_T^{\mathrm{Tama}}$. We start with a quick review of the definition of μ_T^{Tama} in order to introduce all the relevant notation.

5A. *Tamagawa measure.* Let S be an algebraic torus; here we only discuss the setting where S is defined over $\mathbb{Q}$. The character group of S is the free $\mathbb{Z}$-module $X^*(S) = \widehat{S}$ in Ono's notation (we emphasize that this is the lattice of the characters defined over $\overline{\mathbb{Q}}$). If F is any field containing $\mathbb{Q}$, we let $(\widehat{S})_F$ be the subgroup of characters of S which are defined over F.

As usual, we have $S(\mathbb{A})$ and $S(\mathbb{A}_f)$, the points of S with values in, respectively, the ring of adeles and the ring of finite adeles. The (finite) adeles come equipped with the product absolute value $|\cdot|_{\mathbb{A}}$, and we set

$$S(\mathbb{A})^1 = \{s \in S(\mathbb{A}) : |\chi(s)|_{\mathbb{A}} = 1 \text{ for all } \chi \in (\widehat{S})_{\mathbb{Q}}\}.$$

Let F be a Galois extension which splits S. Then the character lattice $X^*(S)$ can be viewed as a Gal$(F/\mathbb{Q})$-module, and this module uniquely determines S up to isomorphism. We denote this representation by σ_S, and let $L(s, \sigma_S) = \prod_\ell L_\ell(s, \sigma_S)$ be the corresponding Artin L-function (see [Bitan 2011] for a modern treatment). Let r be the multiplicity of the trivial representation in σ_S. By definition,

$$\rho_S := \lim_{s \to 1}(s-1)^r L(s, \sigma_S).$$

Let ω be an invariant gauge form on S. (In particular, ω is defined over $\mathbb{Q}$.) Set

$$\omega^{\mathrm{Tama}} = \omega_\infty \prod_\ell L_\ell(1, \sigma_S)\omega_\ell,$$

where ω_ℓ is the invariant volume form on $S(\mathbb{Q}_\ell)$ induced by ω.

By the product formula, as long as ω is defined over $\mathbb{Q}$, none of the global invariants depend on the normalization of ω.

Let $\chi_1, \ldots, \chi_r$ be a basis for $(\widehat{S})_\mathbb{Q}$, and define a map Λ by

$$S(\mathbb{A}) \xrightarrow{\Lambda} (\mathbb{R}_+^\times)^r, \quad x \mapsto (|\chi_1(x)|_\mathbb{A}, \ldots, |\chi_r(x)|_\mathbb{A}).$$

(In the cases of interest, when $S = T^{\mathrm{der}}$ or $S = T$, we have $r = 0$ or $r = 1$, respectively.) Then Λ induces an isomorphism

$$\widetilde{\Lambda} : S(\mathbb{A})/S(\mathbb{A})^1 \xrightarrow{\sim} (\mathbb{R}_+^\times)^r.$$

(Of course, both sides are trivial if S is anisotropic.)

Define $d\tilde{t}$ by

$$d\tilde{t} := \widetilde{\Lambda}^*\left(\prod_{k=1}^r \frac{dt}{t}\right).$$

Let $dS_\mathbb{Q}$ be the counting measure on $S(\mathbb{Q})$. The Tamagawa measure on $S(\mathbb{A})^1$ defined by Ono [1961, (3.5.2)] (taking into account that in our case the base field denoted by k in [Ono 1961] is $\mathbb{Q}$) is the measure μ^{Tama} that makes the following equality true:

$$\rho_S^{-1}\omega^{\mathrm{Tama}} = d\tilde{t}\,\mu^{\mathrm{Tama}}\,dS_\mathbb{Q}. \tag{5-1}$$

The Tamagawa number is defined by

$$\tau_S = \int_{S^1(\mathbb{A})/S(\mathbb{Q})} \mu^{\mathrm{Tama}}.$$

We will also make use of the differential form on S that we denote by ω_S (this notation agrees with that of [Frenkel et al. 2010]). We define

$$\omega_S := \frac{d\chi_1}{\chi_1} \wedge \cdots \wedge \frac{d\chi_d}{\chi_d}, \tag{5-2}$$

where d is the rank of $X^*(S)$. This form is, a priori, defined over $\overline{\mathbb{Q}}$. However, in fact there exists $D \in \mathbb{Q}$ such that $\omega_S/\sqrt{D}$ is defined over $\mathbb{Q}$. (see [Gross and Gan 1999, Corollary 3.7]). Since the product formula $\sqrt{|D|}\prod_\ell \sqrt{|D|_\ell} = 1$ still holds, we can, in fact, use the form ω_S instead of ω in the definition of the Tamagawa measure, even though it is not quite defined over $\mathbb{Q}$. Specifically, we will from now on work with the form

$$\omega_S^{\mathrm{Tama}} = (\omega_S)_\infty \prod_\ell L_\ell(1, \sigma_S)(\omega_S)_\ell \quad \text{on } S(\mathbb{A}). \tag{5-3}$$

We denote the product over the finite primes by $\omega_{S,f}$, i.e., write $\omega_S^{\mathrm{Tama}} = (\omega_S)_\infty\omega_{S,f}$; the form $\omega_{S,f}$ defines a measure on $S(\mathbb{A}_f)$, the set of points of S over the finite adeles.

5B. *The measure μ^{Tama} versus geometric measure.* This section is based on [Frenkel et al. 2010]. We recall that the measure on orbits that we call μ^{geom} is constructed as a quotient of the measure $|\omega_G|$ by the measure $|\omega_A|$ on our space $\mathbb{A}_G$, which is a "naïve version" of the Steinberg–Hitchin base (see [Gordon 2022, §3.7] for a detailed comparison of the space $\mathbb{A}_G$, and the measure on it, with the actual Steinberg–Hitchin base that is used in [Frenkel et al. 2010]).

Consider the measure on T defined by the form ω_T^{Tama} at every place (its only difference from the Tamagawa measure on T is in the global factor ρ_T). For every finite prime ℓ, let $\mu_{\gamma_\ell}^{\mathrm{Tama}}$ be the measure on the orbit of γ_ℓ in $G(\mathbb{Q}_\ell)$ obtained as the quotient of the measure ω^{can} on G that gives the maximal compact subgroup $G(\mathbb{Z}_\ell)$ volume 1, by the measure $|\omega_T^{\mathrm{Tama}}|_\ell$ on T.

The following proposition is an adaptation of the equality (3.31) of [Frenkel et al. 2010] to our setting.

Proposition 5.1 [Frenkel et al. 2010, Proposition 3.29; (3.31)]. *We have*

$$\mu_{\gamma,\ell}^{\mathrm{geom}} = |\eta(\gamma)|_\ell^{-\frac{1}{4}g(g+1)} \sqrt{|D(\gamma)|_\ell} \, \mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell)) L_\ell(1,\sigma_T) \mu_\ell^{\mathrm{Tama}},$$

where $\eta(\gamma)$ is the multiplier of γ.

Proof. For $\gamma \in G^{\mathrm{der}}$, this is equivalent to the relation (3.31) of [Frenkel et al. 2010]; the additional factor $\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))$ on the right appears here because in (3.31) of [Frenkel et al. 2010], the same measure on G needs to be used on both sides of the equation; here we are using the measure $|\omega_G|$ on the left, and the measure $|\omega_G^{\mathrm{can}}| = |\omega_G|/\mathrm{vol}(G(\mathbb{Z}_\ell))$ on the right; so this correction factor is needed. More precisely, the relation (3.30) in [Frenkel et al. 2010] (which we also reproved in Section 4.2.1 of [Achter and Gordon 2017]) asserts that for a semisimple group G,

$$\mu_{\gamma,\ell}^{\mathrm{geom}} = \sqrt{|D(\gamma)|_\ell} \, |\omega_{T\backslash G}|,$$

where $\omega_{T\backslash G}$ is the quotient of the measure ω_G by the measure ω_T on T defined above in Section 5A, which is the same as the measure ω_T in [Frenkel et al. 2010]. Since by definition (and the remark at the end of Section 5A which allows us to use the form ω_T in the definition of the Tamagawa measure), $\omega_{T,\ell}^{\mathrm{Tama}} = L_\ell(1,\sigma_T)\omega_{T,\ell}$, this proves the proposition for $\gamma \in G^{\mathrm{der}}$. For general γ, the factor $|\eta(\gamma)|^{-\frac{1}{4}g(g+1)}$ appears on the right-hand side because we are using the space $\mathbb{A}_G$ instead of the Steinberg–Hitchin base of GSp_{2g}. This factor is calculated by considering the action of the centre of G on all the measure spaces involved. This is explained in detail in Section 3.7 of [Gordon 2022]. $\square$

5C. *Proof of Theorem A.* For convenience, we list here the results we proved above about the ratios ν_ℓ and the relevant orbital integrals:

(1) Corollary 4.7 ($\ell \neq p$) $\nu_\ell([X,\lambda]) = \dfrac{\ell^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} O_{\gamma_0}^{\mathrm{geom}}(\phi_0).$

(2) Corollary 4.12 $\nu_p([X,\lambda]) = q^{-\frac{1}{2}g(g+1)} \dfrac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}).$

Combining these with Proposition 5.1, and observing that for $\ell \neq p$, we have $|\eta(\gamma)|_\ell = |\det(\gamma)|_\ell = 1$, and at p, we have $|\eta(\gamma)|_p = q^{-1}$, we obtain

$$
\begin{aligned}
\nu_p([X, \lambda]) &= q^{-\frac{1}{2}g(g+1)} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_p)} O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}) \\
&= q^{-\frac{1}{2}g(g+1)} q^{\frac{1}{4}g(g+1)} \sqrt{|D(\gamma)|_p} \, \mathrm{vol}_{\omega_G}(G(\mathbb{Z}_p)) L_p(1, \sigma_T) \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_p)} O_{\gamma_0}^{\mathrm{Tama}}(\phi_{q,p}) \\
&= q^{-\frac{1}{4}g(g+1)} \sqrt{|D(\gamma)|_p} L_p(1, \sigma_{T/G}) O_{\gamma_0}^{\mathrm{Tama}}(\phi_{q,p}), \\
\nu_\ell([X, \lambda]) &= \frac{\ell^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_\ell)} O_{\gamma_0}^{\mathrm{geom}}(\phi_\ell) \\
&= \sqrt{|D(\gamma)|_\ell} \, \mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell)) L_\ell(1, \sigma_T) \frac{\ell^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_\ell)} O_{\gamma_0}^{\mathrm{Tama}}(\phi_\ell) \\
&= \sqrt{|D(\gamma)|_\ell} L_\ell(1, \sigma_{T/G}) O_{\gamma_0}^{\mathrm{Tama}}(\phi_\ell).
\end{aligned}
\tag{5-4}
$$

Here the notation $L_\ell(1, \sigma_{T/G})$ stands for $L_\ell(1, \sigma_T)(1 - 1/\ell)$ (including the case $\ell = p$), which agrees with the use of this notation in [Frenkel et al. 2010], and the last equality in both cases follows from (2-5).

Taking a product of these over all primes ℓ, and recalling the product formula for absolute values, we obtain

$$
\prod_\ell \nu_\ell = q^{-\frac{1}{4}g(g+1)} |D(\gamma)|^{-\frac{1}{2}} L(1, \sigma_{T/G}) O_\gamma^{\mathrm{Tama}},
\tag{5-5}
$$

where O_γ^{Tama} stands for the product of orbital integrals in the Langlands–Kottwitz formula, with the measure on each factor given by $\omega_{T, \ell}^{\mathrm{Tama}}$.

Lemma 5.2.
$$
\mathrm{vol}_{\omega_{T,f}^{\mathrm{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A}_f)) = \rho_T \frac{1}{(2\pi)^g} \tau(T).
$$

Proof. We start by emphasizing that while in the definition of the Tamagawa number, one starts with an arbitrary differential form defined over $\mathbb{Q}$, since here we have split off the infinite places, the specific choice of the differential form matters. This choice (dictated by our calculations above, specifically, Proposition 5.1) is the form ω_T of (5-2) (which is not defined over $\mathbb{Q}$ but can still be used, as discussed at the end of Section 5A). From this form, we make the form $\omega_T^{\mathrm{Tama}} = (\omega_T)_\infty \omega_{T,f}$ as in (5-3). We need to identify explicitly the component at the infinite place of the form ω_T^{Tama}, which is the same as the component $(\omega_T)_\infty$ of the differential form ω_T. We have a convenient basis for the character lattice of T; see Section 2B. Define the coordinates $(z_1, \ldots, z_g, \lambda)$ on $T(\mathbb{C})$ such that $T(\mathbb{C}) = \{z_1, \ldots, z_g, \lambda z_1^{-1}, \ldots, \lambda z_g^{-1})\}$, and define the characters $\chi_i(z_1, \ldots, z_g, z_1^{-1}, \ldots, \lambda z_g^{-1}) = z_i$, for $i = 1, \ldots, g$, and the multiplier $\eta(z_1, \ldots, z_g, z_1^{-1}, \ldots, \lambda z_g^{-1}) := \lambda$. (Note that in the Appendix, the character lattice of T is described as a quotient of $\mathbb{Z}^{2g}$ instead, which is convenient for the cohomology computations; we do not use this description here.) We write every element of $T(\mathbb{A})$ as $a = a_f a_\infty$, where a_f has the infinity component 1 and $a_\infty = (1, z_1, \ldots, z_g, \lambda)$ has all the components at the finite places equal to 1. In this notation, $T(\mathbb{A})^1$ is defined by the condition $|z_i| = |\lambda| = \|a_f\|^{-1/g}$. We note that the character η coincides with the map $\widetilde{\Lambda}$ from $T(\mathbb{A})/T(\mathbb{A})^1$ to $\mathbb{R}_+$ defined in Section 5A. Then, by the

definition of μ^{Tama}, it is the volume form on $T(\mathbb{A})^1$ given by

$$\mu^{\text{Tama}} \wedge \frac{d\eta}{\eta} = \frac{d\chi_1(a_f a_\infty)}{\chi_1(a_f a_\infty)} \wedge \cdots \wedge \frac{d\chi_g(a_f a_\infty)}{\chi_g(a_f a_\infty)} \wedge \frac{d\eta}{\eta},$$

and thus its component at ∞ is the form $\mu_\infty^{\text{Tama}} = (dz_1/z_1) \wedge \cdots \wedge (dz_g/z_g)$ on $T(\mathbb{A})_\infty^1 \simeq (S^1)^g$. It is an easy exercise (see [Gordon 2022, Example 2.4]) that the form dz/z gives precisely the arc-length measure on the unit circle. Thus, we get

$$\tau_T = \text{vol}_{\mu^{\text{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A})^1) = \rho_T^{-1} \text{vol}_{\omega_T^{\text{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A})^1) = \rho_T^{-1} \text{vol}_{\omega_T^{\text{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A}_f))(2\pi)^g,$$

which completes the proof. □

Now we can complete the proof of the theorem. By the Langlands–Kottwitz formula (in which we choose the Tamagawa measure on T), we have

$$\widetilde{\#} I([X, \lambda]) = \text{vol}_{\omega_T^{\text{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A}_f)) O_\gamma^{\text{Tama}} = \rho_T \frac{1}{(2\pi)^g} \tau_T O_\gamma^{\text{Tama}}.$$

On the other hand, by (5-5), we have

$$\prod_\ell \nu_\ell = q^{-\frac{1}{4}g(g+1)} |D(\gamma)|^{-\frac{1}{2}} L(1, \sigma_{T/G}) O_\gamma^{\text{Tama}}.$$

It remains to recall that we have defined $\nu_\infty = |D(\gamma)|^{\frac{1}{2}}/(2\pi)^g$, and note that the Euler product for $L(1, \sigma_{T/G})$ is conditionally convergent and equals ρ_T (see [Frenkel et al. 2010, (3.25)]), since $\rho_{\mathbb{G}_m}$ is equal to 1, the residue of the Riemann zeta-function at $s = 1$. Thus,

$$q^{-\frac{1}{4}g(g+1)} \tau_T \nu_\infty \prod_\ell \nu_\ell = \frac{\tau_T}{(2\pi)^g} L(1, \sigma_{T/G}) O_\gamma^{\text{Tama}} = \text{vol}_{\omega_T^{\text{Tama}}}(T(\mathbb{Q}) \backslash T(\mathbb{A}_f)) O_\gamma^{\text{Tama}},$$

which completes the proof.

6. Complements

For the convenience of a hypothetical reader interested in explicit calculations, we collect here some reminders concerning the terms which arise in (1-3).

6A. ν_∞. Recall that we have defined (4-2) $\nu_\infty([X, \lambda])$ as $\sqrt{|D(\gamma_0)|}/(2\pi)^g$, where $D(\gamma_0)$ is the Weyl discriminant $D(\gamma_0) = \prod_{\alpha \in \Phi}(1 - \alpha(\gamma_0))$, the product being over all roots of G. We may relate this to the (polynomial) discriminant of $f_{X/\mathbb{F}_q}(T)$, the characteristic polynomial of Frobenius, as follows.

6A1. *Weyl discriminants.* Explicitly, γ_0 has multiplier $\lambda_0 := \eta(\gamma_0) = q$. Write the (complex) eigenvalues of (a $\mathbb{Q}$-representative of) γ_0 — equivalently, the roots of $f_{X/\mathbb{F}_q}(T)$ — as $(\lambda_1, \ldots, \lambda_g, \lambda_0/\lambda_1, \ldots, \lambda_0/\lambda_g)$. Then

$$D(\gamma) = \prod_{1 \leq i < j \leq g} \delta_{ij} \cdot \prod_{1 \leq i \leq g} \delta_i,$$

where

$$\delta_{ij} = (1 - \lambda_i/\lambda_j)(1 - \lambda_j/\lambda_i)(1 - \lambda_i\lambda_j/\lambda_0)(1 - \lambda_0/(\lambda_i\lambda_j)) \quad \text{and} \quad \delta_i = (1 - \lambda_i^2/\lambda_0)(1 - \lambda_0/\lambda_i^2).$$

Possibly after reordering the conjugate pairs $\{\lambda_i, \lambda_0/\lambda_i\}$, we may and do assume that $\lambda_j = \sqrt{q}\exp(i\theta_j)$ with $0 \leq \theta_j < \pi$. Then

$$\delta_{ij} = (2\cos(\theta_i) - 2\cos(\theta_j))^2 \quad \text{and} \quad \delta_j = 4\sin^2(\theta_j).$$

6A2. *Elliptic curves.* Suppose that $[X, \lambda]$ is an elliptic curve with its canonical principal polarization, say with characteristic polynomial of Frobenius $T^2 - aT + q$. Then $a = 2\sqrt{q}\cos(\theta)$, and $D(\gamma_0) = 4\sin^2(\theta) = 4 - a^2/q$, and

$$v_\infty([X, \lambda]/\mathbb{F}_q) = \frac{1}{2\pi}\sqrt{|D(\gamma_0)|} = \frac{1}{\pi}\sqrt{1 - \frac{a^2}{4q}}.$$

Note that this term is *half* the archimedean term introduced in [Gekeler 2003, (3.3)] (when $q = p$) and [Achter and Gordon 2017, (2-7)]. For purposes of comparison, we summarize this relationship by writing

$$v_\infty([X, \lambda]) = \tfrac{1}{2}v_\infty^{\mathrm{Gek}}([X, \lambda]) = \tfrac{1}{2}v_\infty^{\mathrm{AG}}([X, \lambda]).$$

6A3. *Polynomial discriminants.* To facilitate comparison with [Achter and Williams 2015; Gekeler 2003; Gerhard and Williams 2019], we express $D(\gamma_0)$ in terms of polynomial discriminants. Let $f(T) = f_{X/\mathbb{F}_q}(T)$, and let $f^+(T) = f_{X/\mathbb{F}_q}^+(T)$ be the minimal polynomial of the sum of γ_0 and its adjoint, so that

$$f_{X/\mathbb{F}_q}^+(T) = \prod_{1 \leq j \leq g}(T - (\lambda_j + q/\lambda_j)).$$

Note that $\mathbb{Q}[T]/f^+(T) \cong K^+$, the maximal totally real subalgebra of the endomorphism algebra of X.

Lemma 6.1.
$$\frac{\mathrm{disc}(f(T))}{\mathrm{disc}(f^+(T))} = (-1)^g q^{\frac{1}{2}g(3g-1)}D(\gamma_0).$$

Proof. On one hand,

$$\mathrm{disc}(f(T)) = \prod_{1 \leq i < j \leq g}\alpha_{ij}^2 \prod_{1 \leq i \leq g}\alpha_i^2,$$

where

$$\alpha_{ij} = (\lambda_i - \lambda_j)(\lambda_i - \lambda_0/\lambda_j)(\lambda_0/\lambda_i - \lambda_j)(\lambda_0/\lambda_i - \lambda_0/\lambda_j) \quad \text{and} \quad \alpha_i = (\lambda_i - \lambda_0/\lambda_i).$$

On the other hand,

$$\mathrm{disc}(f^+(T)) = \prod_{1 \leq i < j \leq g}\beta_{ij}^2,$$

where

$$\beta_{ij} = (\lambda_i + \lambda_0/\lambda_i - (\lambda_j + \lambda_0/\lambda_j)).$$

Now use this to evaluate $\mathrm{disc}(f(T))/\mathrm{disc}(f^+(T))$, while bearing in mind that

$$\frac{\alpha_{ij}}{\beta_{ij}^2} = \lambda_0, \quad \frac{\alpha_{ij}}{\delta_{ij}} = \lambda_0^2 \quad \text{and} \quad \frac{\alpha_i^2}{\delta_i} = -\lambda_0. \qquad \square$$

6B. ν_ℓ**.** Gekeler [2003] observed that, for elliptic curves, his product formula essentially computes an L-function; a similar phenomenon has been observed in other contexts as well [Achter and Williams 2015; Gerhard and Williams 2019]. We briefly explain how this relates to (1-3). This detour also has the modest benefit of showing that the right-hand side of (1-3) converges, albeit conditionally.

6B1. *Zeta functions.* We express the zeta function of a number field M as $\zeta_M(s) = \prod_\ell \zeta_{M,\ell}(s)$, where $\zeta_{M,\ell}(s) = \prod_{\lambda|\ell}(1 - N_{M/\mathbb{Q}}(\lambda)^{-s})^{-1}$. For a direct sum $M = \bigoplus_{i=1}^t M_i$ of such fields we write $\zeta_{M,\ell}(s) = \prod_i \zeta_{M_i,\ell}(s)$; the product over all primes yields $\zeta_M(s) = \prod_i \zeta_{M_i}(s)$.

Recall that to a torus $S/\mathbb{Q}$ one associates an Artin L-function $L(s, \sigma_S) = \prod_\ell L_\ell(s, \sigma_S)$. This construction is multiplicative for exact sequences of tori, and for a finite direct sum M of number fields one has $L(s, \sigma_{R_{M/\mathbb{Q}}\mathbb{G}_m}) = \zeta_M(s)$. (It may be worth recalling that $R_{M/\mathbb{Q}}\mathbb{G}_m \cong \bigoplus R_{M_i/\mathbb{Q}}\mathbb{G}_m$.)

If ℓ is unramified in some splitting field for S, then (see [Bitan 2011, 2.8; Voskresenskiǐ 1998, 14.3])

$$\#S(\mathbb{F}_\ell) = \ell^{\dim S} L_\ell(1, \sigma_S)^{-1}.$$

Lemma 6.2. *Suppose that* $\ell \nmid 2p\,\mathrm{disc}(f_{X/\mathbb{F}_q}(T))$. *Then*

$$\nu_\ell([X, \lambda]) = \frac{\zeta_{K,\ell}(1)}{\zeta_{K^+,\ell}(1)}.$$

Proof. By Lemma 4.4,

$$\nu_\ell([X, \lambda]) = \frac{\#\{\gamma \in G(\mathbb{F}_\ell) : \gamma \sim \pi_1(\gamma_0)\}}{\#G(\mathbb{F}_\ell)/\#\mathbb{A}_G(\mathbb{F}_\ell)} = \frac{\#G(\mathbb{F}_\ell)/\#T(\mathbb{F}_\ell)}{\#G(\mathbb{F}_\ell)/(\ell^g \#\mathbb{G}_m(\mathbb{F}_\ell))} = \ell^g \frac{\#\mathbb{G}_m(\mathbb{F}_\ell)}{\#T(\mathbb{F}_\ell)} = \frac{L_\ell(1, \sigma_T)}{L_\ell(1, \sigma_{\mathbb{G}_m})},$$

since $\dim T = g + 1$. Using (2-3), first to see that $L(s, \sigma_T) = L(s, \sigma_{T^{\mathrm{der}}})L(s, \sigma_{\mathbb{G}_m})$ and second to compute $L(s, \sigma_{T^{\mathrm{der}}})$, we recognize the final expression in the above equation to be

$$\frac{\zeta_{K,\ell}(1)}{\zeta_{K^+,\ell}(1)}. \qquad \square$$

Since $\zeta_K(s)$ and $\zeta_{K^+}(s)$ both have a simple pole at $s = 1$, we immediately deduce:

Corollary 6.3. *The right-hand side of* (1-3) *converges conditionally.*

Moreover, up to a finite factor $B([X, \lambda])$, we can express $\widetilde{\#}I([X, \lambda], \mathbb{F}_q)$ in terms of familiar quantities:

Corollary 6.4. *We have*

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_q) = \tau_T \frac{q^{-\frac{1}{4}g(3g-1)}}{(2\pi)^g} \sqrt{\left|\frac{\mathrm{disc}(f)}{\mathrm{disc}(f^+)}\right|} B([X, \lambda]) \lim_{s \to 1^+} \frac{\zeta_K(s)}{\zeta_{K^+}(s)},$$

where

$$B([X, \lambda]) = \prod_{\ell | 2p\,\mathrm{disc}(f)} \frac{\zeta_{K^+,\ell}(1)}{\zeta_{K,\ell}(1)} \nu_\ell([X, \lambda]).$$

6B2. *Elliptic curves.* If $[X, \lambda]$ is an elliptic curve, let χ be the quadratic character associated to the imaginary quadratic field K. Then $K^+ = \mathbb{Q}$, and $\zeta_K(s)/\zeta_{\mathbb{Q}}(s) = L(s, \chi)$.

Now further suppose that $q = p$ and that the Frobenius order is maximal, i.e., that $\mathbb{Z}[T]/f_{X/\mathbb{F}_q}(T) \cong \mathcal{O}_K$. Then Gekeler shows with an explicit calculation that for *each* prime ℓ, $\nu_\ell^{\text{Gek}}([X, \lambda]) = L_\ell(1, \chi)$, and thus $\prod_\ell \nu_\ell^{\text{Gek}}([X, \lambda]) = L(1, \chi)$.

6B3. *Abelian varieties with maximal Frobenius order.* Similarly, suppose $[X, \lambda]$ is an ordinary abelian surface with $\text{End}(X) \otimes \mathbb{Q}$ a cyclic quartic extension of $\mathbb{Q}$, and further suppose that the Frobenius order is maximal. Achter and Williams [2015] define a local term $\nu_\ell^{AW}([X, \lambda])$, and show that $\prod_\ell \nu_\ell^{AW}([X, \lambda]) = \zeta_K(1)/\zeta_{K^+}(1)$. This observation has been extended to certain abelian varieties of prime dimension [Gerhard and Williams 2019, Proposition 8.1].

6C. ν_p. Since the multiplier $\eta(\gamma_0)$ of Frobenius is q, γ_0, while an element of $M(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$, is never an element of $G(\mathbb{Z}_p)$. Nonetheless, if the isogeny class is *ordinary*, it is possible to transfer part of the work in calculating $\nu_p([X, \lambda])$ from $M(\mathbb{Z}_p)$ to $G(\mathbb{Z}_p)$, as follows.

Suppose X is ordinary. Then its p-divisible group splits *integrally* as $X[p^\infty] = X[p^\infty]^{\text{tor}} \oplus X[p^\infty]^{\text{ét}}$. (In general, the slope filtration only exists up to isogeny, as in Lemma 3.5.) Thus, there exists a decomposition $V_{\mathbb{Z}_p} = V_{\mathbb{Z}_p}^{\text{ét}} \oplus V_{\mathbb{Z}_p}^{\text{tor}}$ into maximal isotropic summands stable under γ_0, where $\alpha_0 := \gamma_0|_{V_{\mathbb{Z}_p}^{\text{ét}}} \in \text{End}(V_{\mathbb{Z}_p}^{\text{ét}})$ is invertible; and the polarization induces an isomorphism $V_{\mathbb{Z}_p}^{\text{tor}}$ with the dual of $V_{\mathbb{Z}_p}^{\text{ét}}$, such that $\gamma_0|_{V_{\mathbb{Z}_p}^{\text{tor}}} = q(\alpha_0^\top)^{-1}$. This can also be proved directly through linear algebra. Indeed, let $\beta_0 = q\gamma_0^{-1}$. Then $V_{\mathbb{Z}_p}^{\text{ét}} = \bigcap_n \gamma_0^{\circ n}(V_{\mathbb{Z}_p})$, while $V_{\mathbb{Z}_p}^{\text{tor}} = \bigcap_n \beta_0^{\circ n}(V_{\mathbb{Z}_p})$.

Lemma 6.5. *For n and d sufficiently large and $\gamma \in M(\mathbb{Z}_p/p^n)$, the following conditions are equivalent:*

(a) $\gamma \sim_{M(\mathbb{Z}_p/p^n)_d} \gamma_0 \mod p^n$;

(b) *there exists some $\widetilde{\gamma} \in M(\mathbb{Z}_p)$ such that $\widetilde{\gamma} \mod p^n = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_p)} \gamma_0$;*

(c) γ *stabilizes a decomposition $V_{\mathbb{Z}_p/p^n} \cong V_{\mathbb{Z}_p/p^n}^+ \oplus V_{\mathbb{Z}_p/p^n}^-$ into maximal isotropic subspaces, and there exists an isomorphism $\iota : V_{\mathbb{Z}_p/p^n}^+ \to V_{\mathbb{Z}_p/p^n}^{\text{ét}}$ such that $\iota^*\alpha_0 = \gamma|_{V_{\mathbb{Z}_p/p^n}^+}$.*

Proof. The equivalence of (a) and (b) is Lemma 3.2. For the equivalence of (a) and (c), use the argument above to show that any such γ induces an appropriate decomposition of $V_{\mathbb{Z}_p/p^n}$. $\square$

Therefore, if $\alpha_0 \mod p$ is regular, we obtain a version of Lemma 6.2 at p.

Corollary 6.6. *Suppose $[X, \lambda]$ is ordinary and $\text{ord}_p \text{disc}(f_{X/\mathbb{F}_q}(T)) = e \cdot g(g-1)$. Then*

$$\nu_p([X, \lambda]) = \frac{\zeta_{K,p}(1)}{\zeta_{K^+,p}(1)}.$$

Note that we always have $q^{g(g-1)} | \text{disc}(f_{X/\mathbb{F}_q}(T))$. The case $g = 1$ also follows from the explicit calculation in [Gekeler 2003, Theorem 4.4].

Proof. Define $\epsilon_0 \in \text{Sp}(V_{\mathbb{Z}_p})$ by $\epsilon_0|_{V_{\mathbb{Z}_p}^{\text{ét}}} = \alpha_0$ and $\epsilon_0|_{V_{\mathbb{Z}_p}^{\text{tor}}} = (\alpha_0^\top)^{-1}$.

The argument of Lemma 6.5 shows that, for sufficiently large d and n, both $\#C_{(d,n)}(\gamma_0)$ and $\#C_{(d,n)}(\gamma_0)$ are given by the product of the number of decompositions $V_{\mathbb{Z}_p/p^n} = V_{\mathbb{Z}_p/p^n}^+ \oplus V_{\mathbb{Z}_p/p^n}^-$ into maximal isotropic

subspaces, and the number of $\alpha \in \mathrm{End}(V_{\mathbb{Z}_p/p^n})$ with $\alpha \sim_{\mathrm{End}(V_{\mathbb{Z}_p/p^n})_d} \alpha_0$. In particular, $\#C_{(d,n)}(\gamma_0) = \#C_{(d,n)}(\epsilon_0)$.

The regularity hypothesis implies that $\epsilon_0 \bmod p$ is regular, and the result follows from Lemma 6.2. $\square$

6D. *Explicit examples.*

6D1. $g = 1$. Consider the elliptic curve $E/\mathbb{F}_7$ with affine equation $y^2 = x^3 + x + 1$. Its Frobenius polynomial is $f_E(T) = T^2 - 3T + 7$, which has discriminant -19, a fundamental discriminant. So the order generated by the Frobenius endomorphism is the ring of integers in $K := \mathbb{Q}(\sqrt{-19})$; using Magma, we numerically estimate $\prod_\ell \nu_\ell(E/\mathbb{F}_7) = L\left(1, \left(\frac{-19}{\bullet}\right)\right)$ as ≈ 0.72073. Continuing to work numerically, we obtain that $\nu_\infty(E/\mathbb{F}_7) = \frac{1}{2\pi}\sqrt{4 - \frac{3^2}{7}} \approx 0.2622$. Since $T = \boldsymbol{R}_{K/\mathbb{Q}}\mathbb{G}_m$, we have $\tau_T = 1$, and therefore $\tau_T \sqrt{7} \nu_\infty(E/\mathbb{F}_7) \prod_\ell \nu_\ell(E/\mathbb{F}_7) \approx 0.5000$.

This reflects the easily verified arithmetic statement that the only elliptic curve over $\mathbb{F}_7$ with trace of Frobenius 3 is E itself; and $\mathrm{Aut}(E) \cong \mathcal{O}_K^\times = \{\pm 1\}$, so that the weighted size of this isogeny class is $\widetilde{\#}I([E], \mathbb{F}_7) = \frac{1}{2}$. (In modest contrast, [Gekeler 2003] assigns weight $2/\#\mathrm{Aut}(F)$ to an elliptic curve F; this is reflected in the fact that $\nu_\infty^{\mathrm{Gek}}([E], /\mathbb{F}_7) = 2\nu_\infty([E], \mathbb{F}_7)$.)

6D2. $g = 4$. Consider the 3-Weil polynomial

$$f(T) = T^8 - 6T^7 + 13T^6 - 10T^5 + T^4 - 30T^3 + 117T^2 - 162T + 81.$$

It turns out that there is a unique principally polarized abelian fourfold (X, λ) over $\mathbb{F}_3$ with characteristic polynomial equal to $f(T)$. (This is a single data point in a census of isogeny classes which will soon be integrated into the LMFDB.)

Let $K = \mathbb{Q}[T]/f(T)$. One readily checks that $\mathrm{disc}(f(T))/\mathrm{disc}(K) = 3^{4(4-1)}$, and so $\nu_\ell([X, \lambda]) = \zeta_{K,\ell}(1)/\zeta_{K^+,\ell}(1)$ for all finite ℓ, including $\ell = p$. Again, we numerically estimate $\prod_\ell \nu_\ell([X, \lambda]) = \lim_{s\to 1^+} \zeta_K(s)/\zeta_{K^+}(s) \approx 0.871253$ and $\nu_\infty([X, \lambda]) \approx 0.000111808$. The field K is Galois over $\mathbb{Q}$, with group $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, and the Tamagawa number of the torus T is 2 (see Section A6). Our formula numerically yields

$$\widetilde{\#}I([X, \lambda], \mathbb{F}_3) = 3^3 \tau_T \nu_\infty([X, \lambda], \mathbb{F}_3) \lim_{s\to 1} \frac{\zeta_K(s)}{\zeta_{K^+}(s)} \approx 0.050000.$$

This reflects the fact that the torsion group of $\mathcal{O}_K^\times$, and thus $\mathrm{Aut}([X, \lambda])$, has order 20.

The referee points out that it is possible to take advantage of other recent work on isogeny classes, which appeared essentially simultaneously with or since the first preprint of this work was made available, to independently verify these assertions without recourse to the LMFDB. For example, our characterization of K shows that in fact $K = \mathbb{Q}(\zeta_{20})$; and the discriminant calculation shows that $\mathrm{End}(X) = \mathbb{Z}[\zeta_{20}]$. Since $h(K) = h(K)/h(K^+) = 1$, one can use [Howe 2022, Corollary 1.2] or [Guo et al. 2022, Theorem 1.1] to conclude that $\#I([X, \lambda], \mathbb{F}_3) = 1$.

The state of the art on calculation of Tamagawa numbers has also improved; see [Liang et al. 2021; Rüd 2022b].

6E. *Level structure.* The Langlands–Kottwitz formula (2-1) is actually written for abelian varieties with arbitrary level structure, and thus a version of our main formula is available in the context of abelian varieties with level structure, too.

6E1. *Product formula.* Let $\Gamma \subset G(\hat{\mathbb{Z}}_f^p)$ be an open compact subgroup. There is a notion of principally polarized abelian variety with level Γ structure; let $\mathcal{A}_{g,\Gamma}$ be the corresponding Shimura variety. If $(X, \lambda, \alpha) \in \mathcal{A}_{g,\Gamma}(\mathbb{F}_q)$ is a principally polarized abelian variety with level Γ-structure, then the size of its isogeny class in this category is given by the Kottwitz formula, except that the integrand in the adelic orbital integral is replaced with $\mathbb{1}_\Gamma$.

We make the definition

$$\nu_\ell([X, \lambda, \alpha]) = \lim_{d \to \infty} \lim_{n \to \infty} \frac{\#(C_{(d,n)}(\gamma_0) \cap \pi_n(\Gamma_\ell))}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}.$$

The analogue of Corollary 4.6 holds, and states that there exists $d(\gamma_0)$ such that

$$O_{\gamma_0}^{\mathrm{geom}}(\mathbb{1}_{\Gamma_\ell}) = \lim_{n \to \infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d(\gamma_0),n)}(\gamma_0) \cap \Gamma_\ell)}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

The calculations at p and ∞, as well as the global volume term, are unchanged, and we find that

$$\widetilde{\#}I([X, \lambda, \alpha], \mathbb{F}_q) = q^{\frac{1}{4}g(g+1)} \tau_T \nu_\infty([X, \lambda]) \prod_\ell \nu_\ell([X, \lambda, \alpha]). \tag{6-1}$$

6E2. *Principal level structure.* Fix a prime ℓ_0, and define $\Gamma(\ell_0) = \prod_\ell \Gamma(\ell_0)_\ell$ by

$$\Gamma(\ell_0)_\ell = \begin{cases} G(\mathbb{Z}_\ell) & \ell \neq \ell_0, \\ \ker(G(\mathbb{Z}_{\ell_0}) \to G(\mathbb{Z}_{\ell_0}/\ell_0)) & \ell = \ell_0. \end{cases}$$

Then $\mathcal{A}_{g,\Gamma(\ell_0)}$ is the moduli space of abelian varieties equipped with a full principal level ℓ_0-structure.

For example, to fix ideas, suppose that $g = 1$ and that $\ell_0 \neq p$, and let a satisfy $|a| \leq 2\sqrt{q}$, $p \nmid a$ and $\ell_0 \parallel (a^2 - 4q)$; we consider the set of elliptic curves with characteristic polynomial of Frobenius $f(T) = T^2 - aT + q$. Then some, but not all, elements of the corresponding isogeny class admit a principal level ℓ_0-structure (see, e.g., [Achter and Wong 2013]).

Let (X, λ, α) be an elliptic curve over $\mathbb{F}_q$ with trace of Frobenius a and full level ℓ_0-structure α. We may explicitly compute $\nu_{\ell_0}([X_0, \lambda, \alpha])$ as follows. Let $\chi_{\ell_0} = (\cdot/\ell_0)$ be the quadratic character modulo ℓ_0.

Lemma 6.7.
$$\nu_{\ell_0}([X, \lambda, \alpha]) = \frac{1}{\ell_0^2} \frac{1}{1 - \chi_{\ell_0}(\mathrm{disc}(f)/\ell_0^2)/\ell_0}.$$

Proof. Let $\gamma_0 = \gamma_{X,\mathbb{F}_q,\ell_0}$ be a Frobenius element for X at ℓ_0. By hypothesis, $\gamma_0 = 1 + \ell_0 \beta_0$ for some $\beta_0 \in \mathrm{Mat}_2(\mathbb{Z}_{\ell_0})$. Since ℓ_0^2 is the highest power of ℓ_0 dividing $\mathrm{disc}(f)$, we in fact have $\beta_0 \in \mathrm{GL}_2(\mathbb{Z}_{\ell_0})$, and β_0 is regular mod ℓ_0, i.e., $\pi_1(\beta_0)$ is regular.

Suppose that $\gamma \in \Gamma_{\ell_0}$ satisfies $\gamma \sim_{G(\mathbb{Q}_{\ell_0})} \gamma_0$. Then $\gamma = 1 + \ell_0 \beta$ for some $\beta \in \mathrm{GL}_2(\mathbb{Z}_{\ell_0})$ which is regular mod ℓ_0, and direct calculation shows $\beta \sim_{G(\mathbb{Q}_{\ell_0})} \beta_0$. Lemma 3.1 then shows that $\beta \sim_{G(\mathbb{Z}_\ell)} \beta_0$.

Consequently, for any $d \geq 0$ and any $n \geq 2$, we have bijections between the following sets:

$$\{\gamma \in G(\mathbb{Z}_{\ell_0}/\ell_0^n) : \gamma \sim_{M(\mathbb{Z}_{\ell_0}/\ell_0^n)_d} \pi_n(\gamma_0)\};$$

$$\{\beta \in G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1}) : \beta \sim_{M(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})_d} \pi_{n-1}(\beta_0)\};$$

$$\{\beta \in G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1}) : \beta \sim_{G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} \pi_n(\beta_0)\}.$$

Therefore,

$$\begin{aligned}
\nu_{\ell_0}([X, \lambda, \alpha]) &= \frac{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})/\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G(\mathbb{Z}_{\ell_0}/\ell_0^n)/\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^n)} = \frac{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G(\mathbb{Z}_{\ell_0}/\ell_0^n)} \frac{\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^n)}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} \\
&= \frac{1}{\ell_0^4} \frac{\ell_0^2 \#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} = \frac{1}{\ell_0^2} \frac{\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0)}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0)} = \frac{1}{\ell_0^2} \frac{1}{1 - \chi_{\ell_0}(\mathrm{disc}(f)/\ell_0^2)/\ell_0}. \quad \square
\end{aligned}$$

7. GL_2 reconsidered

In [Achter and Gordon 2017], we essentially treated the $g = 1$ case of the present paper. Unfortunately, a simple algebra error — $\nu_\infty^{\mathrm{AG}}([X, \lambda]) = \frac{2}{\pi}\sqrt{|D(\gamma_0)|}$ (Section 6A2), in spite of the claims of the penultimate displayed equation [Achter and Gordon 2017, p. 20] — masked certain mistakes involving the calculations at p. We take the opportunity to correct these mistakes. The reader pleasantly unaware of these issues with [Achter and Gordon 2017] may simply view the present section as an explication of our technique in the special case where $g = 1$, and thus $G = \mathrm{GL}_2$.

Note that the definition [Achter and Gordon 2017, (2-6)] could have been replaced with a criterion involving characteristic polynomials, e.g.,

$$\nu_p(a, q) = \lim_{n \to \infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}_p/p^n) : f_\gamma \equiv f_{\gamma_0} \bmod p^n\}}{\#G(\mathbb{Z}_p/p^n)/\#A(\mathbb{Z}_p/p^n)}.$$

7A. *Assertions at p.* There are two problematic claims in [Achter and Gordon 2017]:

(1) For the test function $\mathbb{1}_{G(\mathbb{Z}_\ell)}$ at $\ell \neq p$, we have

$$\nu_\ell(a, q) = \frac{\ell^3}{\#\mathrm{SL}_2(\mathbb{F}_\ell)} \mathcal{O}_{\gamma_0}^{\mathrm{geom}}(\mathbb{1}_{G(\mathbb{Z}_\ell)}).$$

It is claimed in [Achter and Gordon 2017, Lemma 3.7] that the same is true for $\ell = p$, where the test function ϕ_q is the characteristic function of $G(\mathbb{Z}_p)\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)G(\mathbb{Z}_p)$.

(2) In [Achter and Gordon 2017, Appendix], revisiting the calculation [Frenkel et al. 2010, (3.30)], we assert that

$$\mu_{\gamma,\ell}^{\mathrm{geom}} = \sqrt{|D(\gamma)|_\ell} \frac{\mathrm{vol}_{|\omega_G|_\ell}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{|\omega_T|_\ell}(T^\circ(\mathbb{Z}_\ell))} \bar{\mu}_{T\backslash G, \ell}. \tag{7-1}$$

This is valid for $\ell \neq p$, but requires correction at p, because our Chevalley–Steinberg map is not exactly the same as the map in [Frenkel et al. 2010].

7B. *From point counts to measure.* In (1), one exploits the fundamental fact that point counts mod ℓ^n converge to volume with respect to the Serre–Oesterlé measure μ^{SO}. In the case where $\ell = p$, however, the ambient space is $\mathrm{Mat}_2(\mathbb{Z}_p)$, rather than (its open subset) $G(\mathbb{Z}_p)$. Thus, the volume of the set V_n of [Achter and Gordon 2017] should be computed with respect to $\mu^{\mathrm{SO}}_{\mathrm{Mat}_2}$, which is the usual measure on the 4-dimensional affine space. We recall that for $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, $d\mu^{\mathrm{SO}}_{\mathrm{GL}_2}(\gamma) = |\det(\gamma)|^{-2} d\mu^{\mathrm{SO}}_{\mathrm{Mat}_2}(\gamma)$.

Moreover, one should be using the invariant measure $dx \wedge (dy/|y|)$ on the Steinberg base $\mathbb{A}_{\mathrm{GL}_2} \cong \mathbb{A}^1 \times \mathbb{G}_m$, rather than the measure pulled back from $\mathbb{A}^1 \times \mathbb{A}^1$. Then the measure of a radius p^{-n} neighbourhood U_n of (a, q) in A is $p^{-2n}/|\det(\gamma_0)| = p^{-2n}/q^{-1}$, and we find

$$v_{p,n}(a, q) = \frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)} \frac{|\det(\gamma_0)|^2 \, \mathrm{vol}_{\mu_{\mathrm{GL}_2}}(V_n(\gamma_0))}{|\det(\gamma_0)| \, \mathrm{vol}_{\mathbb{A}^1 \times \mathbb{G}_m}(U_n)} = |\det(\gamma_0)| \frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_q)$$

$$= q^{-1} \frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_q).$$

(This differs from the assertion of [Achter and Gordon 2017, Lemma 3.7] by a factor of q.)

7C. *From geometric measure to canonical measure.* Since orbital integrals of rational-valued functions with respect to the canonical measure are rational, while $\sqrt{|D(\gamma_0)|_p} = \sqrt{q}$, the assertion of (2) cannot hold at $\ell = p$.

While the relation between the geometric measure and the canonical measure that we rely on is correct for a semisimple group, it needs a correction factor for a reductive group. This part is completely general for all reductive G, and is discussed in detail in [Gordon 2022]; the correct formula is stated in Proposition 5.1. In particular, the correct calculation at p is

$$\mu^{\mathrm{geom}}_{\gamma,p} = |\eta(\gamma)|_p^{-\frac{1}{4}g(g+1)} \sqrt{|D(\gamma)|_p} \frac{\mathrm{vol}_{|\omega_G|_p}(G(\mathbb{Z}_p))}{\mathrm{vol}_{|\omega_T|_p}(T^\circ(\mathbb{Z}_p))} \bar{\mu}_{T \backslash G, p},$$

where $\eta(\gamma)$ is the multiplier of γ.

Appendix

by Wen-Wei Li and Thomas Rüd

We compute the Tamagawa numbers of some anisotropic tori in GSp_{2g} and Sp_{2g} associated with a single Galois field extension (see Section 2B), and present a partial result towards the general case that illustrates the difficulties.

Recall the setup of Section 2B in the case of a single Galois extension. Let $K \supset K^+ \supset \mathbb{Q}$ be a tower of field extensions with K Galois, such that $[K : K^+] = 2$ and $[K^+ : \mathbb{Q}] = g$. We define

$$T^{\mathrm{der}} = \mathrm{Ker}\big(\boldsymbol{R}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{N_{K/K^+}} \boldsymbol{R}_{K^+/\mathbb{Q}}(\mathbb{G}_m)\big) = \boldsymbol{R}_{K^+/\mathbb{Q}} \boldsymbol{R}^{(1)}_{K/K^+}(\mathbb{G}_m) \subset \mathrm{Sp}_{2g},$$

and

$$T = \mathrm{Ker}\big(\mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Q})} \boldsymbol{R}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{(x,y) \mapsto x^{-1} N_{K/K^+}(y)} \boldsymbol{R}_{K^+/\mathbb{Q}}(\mathbb{G}_m)\big) \subset \mathrm{GSp}_{2g}.$$

These fit in the short exact sequence

$$1 \to T^{\mathrm{der}} \to T \to \mathbb{G}_m \to 1.$$

Also, recall that we have been using τ_T (resp. $\tau_{T^{\mathrm{der}}}$) to denote the Tamagawa numbers $\tau_{\mathbb{Q}}(T)$ and $\tau_{\mathbb{Q}}(T^{\mathrm{der}})$. We will show the following.

Proposition A.1. $$\tau_{T^{\mathrm{der}}} = \tau_{K^+}(\boldsymbol{R}^{(1)}_{K/K^+}\mathbb{G}_m) = 2.$$

For the case of T, the result varies with the extension.

Proposition A.2. *Assume that K is a Galois CM-field and K^+ is its maximal totally real subfield. Then we have $\tau_T \le 2$ and*:

- *If g is odd, then $\tau_T = 1$.*
- *If $K/\mathbb{Q}$ is cyclic, then $\tau_T = 1$.*
- *If $g = 2$, then $\tau_T = 1$ when $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ and $\tau_T = 2$ when $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

Additional results and details appear in the second author's thesis [Rüd 2022a]; see also [Rüd 2022b]. We base our approach on the following formula of Ono.

Theorem A.3 [Ono 1963]. *Let $\boldsymbol{T}$ be an algebraic torus defined over a number field F and split over some Galois extension L. Then its Tamagawa number can be computed as*

$$\tau_F(\boldsymbol{T}) = \frac{|H^1(L/F, \boldsymbol{X}^\star(\boldsymbol{T}))|}{|\mathrm{III}^1(\boldsymbol{T})|}.$$

Here $\boldsymbol{X}^\star(\boldsymbol{T})$ denotes the character lattice of $\boldsymbol{T}$. The symbol $\mathrm{III}^1(\boldsymbol{T})$ denotes the corresponding Tate–Shafarevich group defined by

$$\mathrm{III}^i(\boldsymbol{T}) = \mathrm{Ker}\left(H^i(L/F, \boldsymbol{T}) \to \prod_v H^i(L_w/F_v, \boldsymbol{T}) \right), \tag{A-1}$$

where v runs over the primes of F and w is a prime of L with $w|v$.

Our approach is to do the computation on the level of character lattices. A very important consequence of the Tate–Nakayama duality theorem (see [Platonov and Rapinchuk 1994, Theorem 6.10]) is that for a torus $\boldsymbol{T}$ as in the previous theorem, the Pontryagin dual of $\mathrm{III}^1(\boldsymbol{T})$ is isomorphic to $\mathrm{III}^2(\boldsymbol{X}^\star(\boldsymbol{T}))$, so it suffices to compute $|\mathrm{III}^2(\boldsymbol{X}^\star(\boldsymbol{T}))|$.

The proof of Proposition A.1 is given in the next section. The proof of Proposition A.2 occupies Sections A2–A5. In Section A6 we present an example not covered by Proposition A.2. In Section A7 we present a computation for the numerator that illustrates the difficulties that arise for a general torus (not assuming that T is constructed from a single field).

A1. *Computation of $\tau_{T^{\mathrm{der}}}$.* We write the proof of Proposition A.1 using Theorem A.3. Since the Tamagawa number is preserved by restriction of scalars we have $\tau_{T^{\mathrm{der}}} = \tau_{\mathbb{Q}}(\boldsymbol{R}_{K^+/\mathbb{Q}}\boldsymbol{R}^{(1)}_{K/K^+}\mathbb{G}_m) = \tau_{K^+}(\boldsymbol{R}^{(1)}_{K/K^+}\mathbb{G}_m)$. The cohomology of the characters of a norm 1 torus is obtained by a classic computation

that one can see, for instance, in the proof of the Hasse norm theorem in [Platonov and Rapinchuk 1994, Theorem 6.11]. We have

$$\widehat{H}^i(K/\mathbb{Q}, X^\star(T^{\mathrm{der}})) = \widehat{H}^i(K/K^+, R^{(1)}_{K/K^+}\mathbb{G}_m) = \widehat{H}^{i+1}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } i \text{ is odd,} \\ \{0\} & \text{if } i \text{ is even.} \end{cases}$$

In particular, $|\mathrm{III}^1(T^{\mathrm{der}})| = |\mathrm{III}^2(X^\star(T^{\mathrm{der}}))| \le |H^2(X^\star(T^{\mathrm{der}}))| = 1$. We conclude $\tau_{T^{\mathrm{der}}} = \frac{2}{1} = 2$.

This proves Proposition A.1.

A2. *Computation of the first cohomology group of the character lattice.* From now on, we focus on the proof of Proposition A.2, and therefore we will assume that K is a CM-field with K^+ its maximal totally real subfield. Let ι be the nontrivial element of $\mathrm{Gal}(K/K^+)$, and let Γ and Γ^+ denote respectively the Galois groups of $K/\mathbb{Q}$ and $K^+/\mathbb{Q}$. Note that K^+ is indeed Galois over $\mathbb{Q}$ by virtue of K being a CM-field. The torus T arises as the subtorus of $R_{K/\mathbb{Q}}(\mathbb{G}_m)$ with the set of $\mathbb{Q}$-points consisting of elements $x \in K^\times$ such that $x\iota(x) \in \mathbb{Q}$, and $T^{\mathrm{der}}(\mathbb{Q})$ is the set of elements $x \in K^\times$ such that $x\iota(x) = 1$. We have the following exact sequence of finite groups:

$$1 \to \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \Gamma^+ \to 1. \tag{A-2}$$

For each $\sigma \in \Gamma^+$ fix a preimage $\hat{\sigma} \in \Gamma$. We get the description of $X^\star(R_{K/\mathbb{Q}}(\mathbb{G}_m)) = \mathbb{Z}[\Gamma]$ as the set of $\mathbb{Z}$-linear combinations of $\hat{\sigma}$ and $\hat{\sigma}\iota$ with $\sigma \in \Gamma^+$.

The embedding of T in $R_{K/\mathbb{Q}}(\mathbb{G}_m)$ gives us a surjective map $X^\star(R_{K/\mathbb{Q}}(\mathbb{G}_m)) \to X^\star(T)$. For $\chi = \sum_{\sigma \in \Gamma^+} a_\sigma \hat{\sigma} + \sum_{\sigma \in \Gamma^+} b_\sigma \hat{\sigma}\iota \in X^\star(R_{K/\mathbb{Q}}(\mathbb{G}_m))$ and $t \in T(\mathbb{Q})$, we have

$$\begin{aligned} \chi(t) &= \prod_{\sigma \in \Gamma^+} \hat{\sigma}(t)^{a_\sigma} \prod_{\sigma \in \Gamma^+} \hat{\sigma}(\iota(t))^{b_\sigma} \\ &= \prod_{\sigma \in \Gamma^+} \hat{\sigma}(t)^{a_\sigma} \prod_{\sigma \in \Gamma^+} \hat{\sigma}(\lambda t^{-1})^{b_\sigma} \quad \text{(where } t\iota(t) = \lambda \in \mathbb{Q}^\times) \\ &= \lambda^{\sum_{\sigma \in \Gamma^+} b_\sigma} \prod_{\sigma \in \Gamma^+} \hat{\sigma}(t)^{a_\sigma - b_\sigma}. \end{aligned}$$

We get the descriptions

$$X^\star(T) = \mathbb{Z}[\Gamma] / \left\{ \sum_{\sigma \in \Gamma^+} a_\sigma \hat{\sigma} + \sum_{\sigma \in \Gamma^+} b_\sigma \hat{\sigma}\iota : a_\sigma = b_\sigma \text{ for } \sigma \in \Gamma^+ \text{ and } \sum_{\sigma \in \Gamma^+} b_\sigma = 0 \right\} \tag{A-3}$$

$$= \mathbb{Z}[\Gamma] / L, \tag{A-4}$$

where $L = \left\{ \sum_{\sigma \in \Gamma^+} a_\sigma \hat{\sigma}(1 + \iota) : \sum_{\sigma \in \Gamma^+} a_\sigma = 0 \right\}$. For T^{der}, we have $\lambda = 1$, so we recover

$$X^\star(T^{\mathrm{der}}) = \mathbb{Z}[\Gamma] / \{ x = \iota(x) \} = \mathbb{Z}[\widehat{\Gamma^+}] \otimes \mathbb{Z}[\iota] / \langle 1 + \iota \rangle = X^\star(R_{K^+/\mathbb{Q}} R^{(1)}_{K/K^+}(\mathbb{G}_m)). \tag{A-5}$$

In order to compute $H^1(K/\mathbb{Q}, X^\star(T))$ we use the inflation-restriction exact sequence, which one can find in [Gille and Szamuely 2006, Proposition 3.3.14, p. 65]. To simplify notation, let $\Lambda = X^\star(T) = \mathbb{Z}[\Gamma]/L$ as in (A-3).

The inflation-restriction exact sequence associated with the short exact sequence (A-2) takes the form

$$0 \to H^1(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \to H^1(\Gamma, \Lambda) \to H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda)^{\Gamma^+} \to H^2(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \to H^2(\Gamma, \Lambda). \qquad \text{(A-6)}$$

Lemma A.4. *The sequence* (A-6) *can be rewritten as*

$$0 \to 0 \to H^1(\Gamma, \Lambda) \to (\mathbb{Z}/2\mathbb{Z})^{\frac{1}{2}(1+(-1)^g)} \to \Gamma^{+\mathrm{ab}} \to H^2(\Gamma, \Lambda). \qquad \text{(A-7)}$$

In particular, $\tau_T \le |H^1(\Gamma, X^\star(T))| \le 2$.

Proof. Let $x \in \mathbb{Z}[\Gamma]$, and let $[x]$ denote its class in Λ. Clearly $[x]$ is fixed by ι if and only if $x - \iota x \in L$, and since every element of L is fixed by ι, then so must $x - \iota x$ be, which forces $x = \iota x$. Therefore,

$$\Lambda^{\mathbb{Z}/2\mathbb{Z}} = \left\{ \sum_{\sigma \in \Gamma^+} a_\sigma \hat\sigma (1 + \iota) \right\} \Big/ \left\langle \sum_{\sigma \in \Gamma^+} a_\sigma \hat\sigma (1+\iota) : \sum_{\sigma \in \Gamma^+} a_\sigma = 0 \right\rangle \cong \mathbb{Z}[\Gamma^+]/I,$$

where I is the augmentation ideal of $\mathbb{Z}[\Gamma^+]$, i.e., the subspace of sum-zero vectors. Further, observe that $\mathbb{Z}[\Gamma^+]/I \cong \mathbb{Z}$ as Γ^+-modules, where $\mathbb{Z}$ has trivial Γ^+-action (by definition of I). We get that

$$H^1(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \cong H^1(\Gamma^+, \mathbb{Z}) = \mathrm{Hom}(\Gamma^+, \mathbb{Z}) = \{0\}.$$

Also, using the sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0, \qquad \text{(A-8)}$$

since the middle term is uniquely divisible hence cohomologically trivial, one has

$$H^2(\Gamma^+, \mathbb{Z}) \cong H^1(\Gamma^+, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma^+, \mathbb{Q}/\mathbb{Z});$$

the last term is noncanonically isomorphic to Γ^{+ab}.

The only term left to compute is $H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda)^{\Gamma^+}$. As a $\mathbb{Z}/2\mathbb{Z}$-module, we can write $\Lambda = \mathbb{Z}^g \oplus \mathbb{Z}^g/L$, where $L = \{(a, a) : a = (a_1, \ldots, a_g) \text{ such that } \sum a_i = 0\}$, and $\mathbb{Z}/2\mathbb{Z}$ acts as $(a, b) \mapsto (b, a)$. Therefore,

$$0 \to L \to \mathbb{Z}^g \oplus \mathbb{Z}^g = \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]^g \to \Lambda \to 0.$$

Since the middle term is cohomologically trivial as a $\mathbb{Z}/2\mathbb{Z}$-module, and $\mathbb{Z}/2/Z$ acts trivially on L,

$$H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda) \cong H^2(\mathbb{Z}/2\mathbb{Z}, L) \cong \widehat{H}^0(\mathbb{Z}/2\mathbb{Z}, L) \cong L/2L.$$

To compute $L/2L$, we view L as a submodule of $\mathbb{Z}^g$ of zero-sum elements. Recall that by construction of L as a group algebra, Γ^+ acts transitively on L.

Let $\boldsymbol{a} = (a_1, \ldots, a_g) \in L$. We want to compute $(L/2L)^{\Gamma^+}$, and for that, we reason on the parity of a_i's.

- If all a_i are even, then $\boldsymbol{a} = 2\boldsymbol{a}'$ and $\boldsymbol{a}' \in L$, and hence $\boldsymbol{a} \in 2L$.

- If $\boldsymbol{a}$ has a_i even and a_j odd, considering a permutation $\sigma \in \Gamma^+$ sending the i-th coordinate to the j-th, we have that the j-th coordinate of $\boldsymbol{a} - \sigma\boldsymbol{a}$ is $a_j - a_i$, which is odd, and hence $\boldsymbol{a} - \sigma\boldsymbol{a} \notin 2L$.

- The last case to consider is when all a_i are odd. In that case, $\sum_i a_i$ has the same parity as g, so $\boldsymbol{a} \in L$ can only happen if g is even. One can prove that every element of $L/2L$ has a representative of the form $\boldsymbol{a} = (a_1, \ldots, a_g) \in \mathbb{Z}^g$ with $\sum_i a_i = 0$ and $|a_i| \le 1$. When g is even and all a_i are odd, the only

possible such elements are vectors with half the coordinates being -1 and the other half 1. Moreover all such vectors are in the same coset of $2L$ (one can permute the ± 1 coordinates by adding ± 2).

This shows that $(L/2L)^{\Gamma^+}$ contains no nontrivial element when g is odd, and only one when g is even, and concludes the proof. $\qquad\square$

Corollary A.5. *When g is odd, one has $H^1(K/\mathbb{Q}, X^\star(T)) = \{0\}$ and*

$$H^2(K/\mathbb{Q}, X^\star(T)) \cong \Gamma^{+\mathrm{ab}}.$$

In particular, $H^2(K/\mathbb{Q}, X^\star(T))$ has odd order, and so does $\mathrm{III}^1(T)$, since it is dual to $\mathrm{III}^2(X^\star(T)) \subset H^2(K/\mathbb{Q}, X^\star(T))$.

Proof. The first equality comes directly from the sequence (A-7).

For the second equality, taking duals of the exact sequence (2-3), one gets

$$0 \to \mathbb{Z} \to X^\star(T) \to X^\star(T^{\mathrm{der}}) \to 0.$$

The cohomology of this sequence gives us

$$H^1(\Gamma, X^\star(T)) \to H^1(\Gamma, X^\star(T^{\mathrm{der}})) \to H^2(\Gamma, \mathbb{Z}) \to H^2(\Gamma, X^\star(T)) \to H^2(\Gamma, X^\star(T^{\mathrm{der}})).$$

We computed the cohomology $H^i(\Gamma, X^\star(T^{\mathrm{der}})) = H^i(\mathbb{Z}/2\mathbb{Z}, X^\star(R^{(1)}_{K/K^+}\mathbb{G}_m))$ in Section A1.

We can substitute $H^1(\Gamma, X^\star(T)) = \{0\} = H^2(\Gamma, X^\star(T^{\mathrm{der}}))$, $H^2(\Gamma, \mathbb{Z}) \cong H^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong \Gamma^{ab}$, and $H^2(\Gamma, X^\star(T^{\mathrm{der}})) \cong \mathbb{Z}/2\mathbb{Z}$, which gives us

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma^{ab} \to H^2(\Gamma, X^\star(T)) \to 0,$$

as desired. $\qquad\square$

Proposition A.6. *When the sequence (A-2) splits,*

$$H^1(K/\mathbb{Q}, X^\star(T)) = \begin{cases} \{0\} & \text{if } g \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } g \text{ is even.} \end{cases}$$

In particular, this gives an alternate proof of the triviality of $H^1(K/\mathbb{Q}, X^\star(T))$ whenever g is odd.

Proof. Since (A-2) splits, one can write the inflation-restriction exact sequence associated with the short exact sequence

$$1 \to \Gamma^+ \to \Gamma \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

This gives us

$$0 \to H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) \to H^1(\Gamma, \Lambda) \to H^1(\Gamma^+, \Lambda)^{\mathbb{Z}/2\mathbb{Z}} \to H^2(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) \to H^2(\Gamma, \Lambda). \quad \text{(A-9)}$$

Since the sequence (A-2) splits, we have $\mathbb{Z}[\Gamma] = \mathbb{Z}[\Gamma^+] \otimes \mathbb{Z}[\iota]$ as a Γ-module. We have $\Lambda = \mathbb{Z}[\Gamma^+] \otimes \mathbb{Z}[\iota]/I \otimes (1+\iota)$, where I is the augmentation ideal of $\mathbb{Z}[\Gamma^+]$. Because $\mathbb{Z}[\Gamma^+]$ is an induced module, and $I \otimes (1+\iota) \cong I$ as a Γ^+-module, we get $\widehat{H}^i(\Gamma^+, \Lambda) = \widehat{H}^{i+1}(\Gamma^+, I)$. Now since $\mathbb{Z} = \mathbb{Z}[\Gamma^+]/I$ with $\mathbb{Z}$ seen as a trivial module, the same argument yields $\widehat{H}^i(\Gamma^+, \Lambda) \cong \widehat{H}^i(\Gamma^+, \mathbb{Z})$. In particular, $H^1(\Gamma^+, \Lambda) = \{0\}$, so the sequence (A-9) gives an isomorphism $H^1(\Gamma, \Lambda) \cong H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+})$.

Direct computations using that $\overset{+}{\sigma} := \sum_{\sigma \in \Gamma^+} \sigma$ spans the set of Γ^+-fixed elements of $\mathbb{Z}[\Gamma^+]$ give us that $\{1 \otimes (1 + \iota), \overset{+}{\sigma} \otimes \iota\}$ is a $\mathbb{Z}$-basis for Λ^{Γ^+}, on which ι acts via $\left(\begin{smallmatrix} 1 & g \\ 0 & -1 \end{smallmatrix}\right)$. Identifying the space with $\mathbb{Z}^2$ we can compute cocycles and coboundaries. Coboundaries are of the form $a_\iota = (-gb, 2b)$ for $b \in \mathbb{Z}$. Cocycles are of the form $a_\iota = (a, b)$ with $2a + gb = 0$. Thus, if b is even then it is a coboundary, if b is odd then g cannot be odd, and so we only get a nontrivial cocycle with g even and b odd. The difference of two nontrivial cocycles has an even second entry, so it is a coboundary. This proves

$$H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) = \begin{cases} \{0\} & \text{if } g \text{ is odd}, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } g \text{ is even}, \end{cases}$$

as desired.

For the last assertion in the proposition, when g is odd, by the Schur–Zassenhaus theorem (see [Rotman 1995, Theorem 7.41]) the sequence (A-2) splits and we get our result immediately. □

A3. *Case g odd.* Now we are ready to show:

Lemma A.7. *If g is odd, then $\tau_T = \frac{1}{1} = 1$.*

Proof. Using Corollary A.5 we have that $|H^1(K/\mathbb{Q}, X^\star(T))| = 1$, and $\text{III}^1(T)$ has odd order. To show that $\text{III}^1(T)$ is trivial, it suffices to show it is 2-torsion. The cohomology of the sequence (2-3) yields

$$H^1(K/\mathbb{Q}, T^{\text{der}}) \to H^1(K/\mathbb{Q}, T) \to H^1(K/\mathbb{Q}, \mathbb{G}_m) = 1,$$

where the right equality holds by Hilbert 90. So we have a surjection of $H^1(K/\mathbb{Q}, T^{\text{der}})$ onto $H^1(K/\mathbb{Q}, T)$. We claim that $H^1(K/\mathbb{Q}, T)$ is 2-torsion; it suffices so show that $H^1(K/\mathbb{Q}, T^{\text{der}})$ is.

Taking the cohomology of the sequence

$$1 \to \mathbf{R}^{(1)}_{K/K^+} \mathbb{G}_m \to \mathbf{R}_{K/K^+} \mathbb{G}_m \xrightarrow{N_{K/K^+}} \mathbb{G}_m \to 1,$$

where the middle term is cohomologically trivial, we have $H^1(K/K^+, \mathbf{R}^{(1)}_{K/K^+} \mathbb{G}_m) \cong \widehat{H}^0(K/K^+, \mathbb{G}_m) = (K^+)^\times / N_{K/K^+} K^\times$.

This gives us

$$H^1(K/\mathbb{Q}, T^{\text{der}}) = H^1(K/\mathbb{Q}, \mathbf{R}_{K^+/\mathbb{Q}} \mathbf{R}^{(1)}_{K/K^+} \mathbb{G}_m) = H^1(K/K^+, \mathbf{R}^{(1)}_{K/K^+} \mathbb{G}_m) \cong (K^+)^\times / N_{K/K^+} K^\times.$$

This group is 2-torsion, hence so is $H^1(K/\mathbb{Q}, T)$ and $\text{III}^1(T)$ is a subgroup of the latter. We can conclude that $\text{III}(T)$ is a 2-torsion group of odd order, hence it is trivial.

We can conclude using Theorem A.3 that $\tau_T = \frac{1}{1} = 1$.

Note that one need not use Corollary A.5 to know that $\text{III}^1(T)$ has odd order and hence is trivial. Indeed, given that the extension K/K^+ is quadratic, by the Chebotarev density theorem, we know that there is a prime $\mathfrak{p} \in K^+$ inert in the extension K/K^+. Since $\mathfrak{p}$ is stable under ι, which is of order 2, then its decomposition group $\Gamma(\mathfrak{p})$ has even order, and therefore odd index in Γ. Now it suffices to look at the restriction-corestriction sequence $H^1(\Gamma, T) \to H^1(\Gamma(\mathfrak{p}), T) \to H^1(\Gamma, T)$. The composition of the two maps is just multiplication by $n = [\Gamma : \Gamma(\mathfrak{p})]$. By definition of $\text{III}^1(T)$, this subgroup of $H^1(\Gamma, T)$ is killed by the restriction map, hence it is n-torsion, and we know n is odd, as desired. □

A4. *The case $K/\mathbb{Q}$ cyclic.*

Lemma A.8. *When K is cyclic, we have $\tau(T) = \frac{1}{1} = 1$. In particular, this holds when $K^+/\mathbb{Q}$ is cyclic of odd order.*

Proof. Write $X^\star(T) = \mathbb{Z}[\Gamma]/L$ as in (A-3).

By virtue of K being cyclic, the Tate cohomology is 2-periodic, so

$$H^1(K/\mathbb{Q}, X^\star(T)) \cong H^2(K/\mathbb{Q}, L) \cong \widehat{H}^0(K/\mathbb{Q}, L).$$

Since L has trivial ι action, we can see it as the augmentation ideal of $\mathbb{Z}[\Gamma^+]$, which has no Γ^+-fixed point, as any augmentation ideal. In particular it has no Γ-fixed point and so $\widehat{H}^0(K/\mathbb{Q}, L) = \{0\}$.

Again using the fact that K is cyclic, we get that $\text{III}^1(T)$ is trivial. Indeed, by the Chebotarev density theorem, every cyclic extension has a prime $p \in \mathbb{Z}$ that will stay inert, and therefore $\Gamma(p) = \Gamma$, where $\Gamma(p)$ is the corresponding decomposition group. Therefore, the map in the definition of $\text{III}^1(T)$ is injective.

We can conclude by Theorem A.3 that $\tau_T = \frac{1}{1} = 1$. $\qquad\square$

A5. *Case $g = 2$.*

Lemma A.9. *When $g = 2$ we have $\tau_T = 1$ if Γ is cyclic and $\tau_T = 2$ otherwise.*

Proof. The first case is a consequence of Lemma A.8. If Γ isn't cyclic, the only possibility is $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^2$.

In that case, Proposition A.6 gives $H^1(\Gamma, X^\star(T)) \cong \mathbb{Z}/2\mathbb{Z}$. Concerning the Tate–Shafarevich group, Cortella [1997] showed that $\text{III}^1(T) = \{0\}$ if $g < 4$.

Alternatively, since Γ is abelian, every proper cyclic subgroup appears as decomposition group. In this specific case, it is a consequence of the Chinese remainder theorem and quadratic reciprocity. Using SageMath computations (see below), we obtain that the map

$$H^2(\Gamma, X^\star(T)) \to H^2(\mathbb{Z}/2\mathbb{Z} \times \{0\}, X^\star(T)) \oplus H^2(\{0\} \times \mathbb{Z}/2\mathbb{Z}, X^\star(T))$$

is injective, and therefore $\text{III}^1(T) = 0$. $\qquad\square$

A6. *Computing the Tamagawa numbers with SageMath.*

The second author implemented methods in SageMath to deal with algebraic tori through their character lattices. Those methods should eventually be added to SageMath in a future release.

Here we briefly describe the computation of the Tamagawa number which arises in Section 6D2, where $K = \mathbb{Q}[T]/f(T)$ for

$$f(T) = T^8 - 6T^7 + 13T^6 - 10T^5 + T^4 - 30T^3 + 117T^2 - 162T + 81.$$

We have $\Gamma = \text{Gal}(K/\mathbb{Q}) = \mathbb{Z}/4 \oplus \langle \iota \rangle$, where ι denotes the complex involution.

Nakayama duality lets us compute the Tamagawa number as a function of the character lattice:

$$\tau_{\mathbb{Q}}(T) = \frac{|H^1(\mathbb{Q}, X^\star(T))|}{|\text{III}^2(X^\star(T))|}.$$

Let Λ denote $X^\star(T)$. We build Λ in SageMath by inducing the trivial lattice $\mathbb{Z} = X^\star(\mathbb{G}_m)$ to Γ, build the sublattice of zero-sum elements of ι-fixed points, and take the quotient of the former by the latter.

We can compute the first cohomology group by computing cocycles as solutions of linear equations in $\Lambda^{|\Gamma|}$. However, for this example, Proposition A.6 gives us $H^1(\Gamma, \Lambda) = 2$. For the denominator, we build a method that, given $\mathcal{H}$ a collection of subgroups of Γ, and a Γ-lattice L, computes

$$\mathrm{III}^1_{\mathcal{H}}(L) = \mathrm{Ker}\left(H^1(\Gamma, L) \to \bigoplus_{\Delta \in \mathcal{H}} H^1(\Delta, L) \right)$$

by checking which cocycles restrict to coboundaries on all $\Delta \in \mathcal{H}$.

Consider the embedding $\varphi : \Lambda \to \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda$ (with Γ-action on the left component) via $a \mapsto \sum_{g \in \Gamma} g \otimes g^{-1} a$. We build

$$\Lambda' = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda / \varphi(\Lambda).$$

Since $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda$ is induced, it is cohomologically trivial, so $\widehat{H}^i(\Gamma, \Lambda) = \widehat{H}^{i-1}(\Gamma, \Lambda')$ for all $i \in \mathbb{Z}$. Consequently, if every element of $\mathcal{H}$ arises as a decomposition group, we have

$$\mathrm{III}^1_{\mathcal{H}}(\Lambda') \supset \mathrm{III}^1(\Lambda') = \mathrm{III}^2(\Lambda).$$

We take $\mathcal{H}$ to be the list of cyclic subgroups of Γ. They all arise as decomposition groups. Using our new SageMath package, we get $|\mathrm{III}^1_{\mathcal{H}}(\Lambda')| = 1 \geq \mathrm{III}^2(\Lambda) \geq 1$, and hence

$$\tau(T) = \frac{|H^1(\Gamma, \Lambda)|}{|\mathrm{III}^2(\Lambda)|} = \frac{2}{1} = 2.$$

A7. *The numerator in Ono's formula: general case.* We give some indications for the general case in which T is described by a CM-algebra $K = \bigoplus_{i=1}^t K_i$, each K_i being a CM-field. The Rosati involution on K is still denoted as ι, with fixed subalgebra $K^+ = \bigoplus_{i=1}^t K_i^+$. In this section we denote by Γ the absolute Galois group of $\mathbb{Q}$. Our modest aim is to understand $|H^1(\Gamma, X^\star(T))|$ through Kottwitz's isomorphism (see [Kottwitz 1984b, (2.4.1) and §2.4.3])

$$H^1(\Gamma, X^\star(T)) \cong \pi_0(\widehat{T}^\Gamma),$$

where $\widehat{T}$ is the dual $\mathbb{C}$-torus. This isomorphism is valid for all tori.

To describe $X^\star(T)$, we first write T as

$$T = (\mathbb{G}_m \times T^{\mathrm{der}})/\{(z, z) : z \in \mu_2\}, \quad \mu_2 := \{\pm 1\}.$$

Choose a subset $\Phi = \bigsqcup_{i=1}^t \Phi_i$ of $\mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K, \overline{\mathbb{Q}})$, such that $\Phi_i \subset \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K_i, \overline{\mathbb{Q}})$ and

$$\mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K_i, \overline{\mathbb{Q}}) = \Phi_i \sqcup \Phi_i \iota$$

for all $i = 1, \ldots, t$. Note that $|\Phi| = g$. It is well known that

$$X^\star(T^{\mathrm{der}}) = \bigoplus_{\phi \in \Phi} \mathbb{Z}\epsilon_\phi$$

for some basis $\{\epsilon_\phi\}_{\phi \in \Phi}$. Then Γ permutes $\{\pm\epsilon_\phi\}_{\phi \in \Phi}$ by

$$\sigma \epsilon_\phi = \begin{cases} \epsilon_\psi & \text{if } \sigma\phi = \psi \in \Phi, \\ -\epsilon_\psi & \text{if } \sigma\phi = \psi\iota \in \Phi\iota, \end{cases} \qquad \phi, \psi \in \Phi. \qquad (A\text{-}10)$$

The inclusion $\mu_2 \hookrightarrow T^{\text{der}}$ corresponds to the map

$$X^\star(T^{\text{der}}) \twoheadrightarrow X^\star(\mu_2) = \mathbb{Z}/2\mathbb{Z}, \qquad \sum_{\phi \in \Phi} x_\phi \epsilon_\phi \mapsto \sum_{\phi \in \Phi} x_\phi \mod 2.$$

Write $X^\star(\mathbb{G}_m) = \mathbb{Z}\eta$, where η is the standard generator. Applying Cartier duality to the exact sequence

$$1 \to \mu_2 \to \mathbb{G}_m \times T^{\text{der}} \to T \to 1,$$

we obtain

$$X^\star(T) = \left\{ t\eta + \sum_{\phi \in \Phi} x_\phi \epsilon_\phi : t + \sum_\phi x_\phi \in 2\mathbb{Z} \right\} \subset X^\star(\mathbb{G}_m) \oplus X^\star(T^{\text{der}})$$

and a basis of $X^\star(T)$,

$$\{2\eta\} \sqcup \{\eta + \epsilon_\phi : \phi \in \Phi\}.$$

The element 2η is surely Γ-invariant. On the other hand, for $\phi, \psi \in \Phi$ and $\sigma \in \Gamma$, we derive from (A-10) that

$$\sigma(\eta + \epsilon_\phi) = \begin{cases} \eta + \epsilon_\phi & \text{if } \sigma\phi = \psi \in \Phi, \\ 2\eta - (\eta + \epsilon_\psi) & \text{if } \sigma\phi = \psi\iota \in \Phi\iota. \end{cases} \qquad (A\text{-}11)$$

The Γ-action on $\widehat{T} := X^\star(T) \otimes \mathbb{C}^\times \cong \mathbb{C}^\times \times (\mathbb{C}^\times)^\Phi$ is thus

$$\sigma \cdot (z, \overbrace{1, \ldots, \underbrace{w}_{\phi}, \ldots, 1}^{\Phi}) = \begin{cases} (z, 1, \ldots, \underbrace{w}_{\psi}, \ldots, 1) & \text{if } \sigma\phi = \psi \in \Phi, \\ (zw, 1, \ldots, \underbrace{w^{-1}}_{\psi}, \ldots, 1) & \text{if } \sigma\phi = \psi\iota \in \Phi\iota. \end{cases}$$

Recall from the description of $X^\star(T^{\text{der}})$ that $\widehat{T^{\text{der}}}$ can be identified with $(\mathbb{C}^\times)^\Phi$. Note that $(\widehat{T^{\text{der}}})^\Gamma$ is finite since T^{der} is anisotropic.

Lemma A.10. *There is a canonical isomorphism $(\widehat{T^{\text{der}}})^\Gamma \cong \mu_2^t$ characterized as follows. For $(a_i)_{i=1}^t \in \mu_2^t$, the corresponding $\hat{t} = (\hat{t}_\phi)_{\phi \in \Phi} \in (\widehat{T^{\text{der}}})^\Gamma$ is specified by*

$$\phi \in \Phi_i \implies \hat{t}_\phi = a_i \quad \text{for all } 1 \le i \le t.$$

Proof. Shapiro's lemma reduces the computation of $(\widehat{T^{\text{der}}})^\Gamma$ or $H^1(\Gamma, X^\star(T))$ to the easy case $t = 1$ over the base field K^+. $\qquad\square$

By dualizing $1 \to T^{\text{der}} \to T \to \mathbb{G}_m \to 1$ into $1 \to \mathbb{C}^\times \to \widehat{T} \to \widehat{T_0} \to 1$, then taking Γ-invariants, we obtain the exact sequence

$$1 \to \mathbb{C}^\times \to \widehat{T}^\Gamma \to (\widehat{T^{\text{der}}})^\Gamma.$$

It induces

$$\pi_0(\widehat{T}^\Gamma) = \widehat{T}^\Gamma/\mathbb{C}^\times \hookrightarrow (\widehat{T^{\text{der}}})^\Gamma = \pi_0((\widehat{T^{\text{der}}})^\Gamma).$$

For each $\sigma \in \Gamma$, set $\Phi(\sigma) := \{\phi \in \Phi : \sigma\phi \notin \Phi\}$. For each i, set $\Phi_i(\sigma) := \Phi(\sigma) \cap \Phi_i$.

Proposition A.11. *For any $(a_i)_i \in \mu_2^t$, the corresponding element $\hat{t} \in (\widehat{T^{\mathrm{der}}})^\Gamma$ belongs to the image of $\widehat{T}^\Gamma/\mathbb{C}^\times$ if and only if*

$$A(a_1, \ldots, a_t; \sigma) := \sum_{\substack{1 \le i \le t \\ a_i = -1}} |\Phi_i(\sigma)| \in 2\mathbb{Z} \quad \text{for all } \sigma \in \Gamma.$$

Proof. Identify $\widehat{T^{\mathrm{der}}}$ with $(\mathbb{C}^\times)^\Phi$. Identify $\widehat{T}$ with $\mathbb{C}^\times \times (\mathbb{C}^\times)^\Phi$ using the basis $\{2\eta\} \sqcup \{\eta + \epsilon_\phi : \phi \in \Phi\}$ of $X^\star(T)$; the homomorphism $\widehat{T} \to \widehat{T^{\mathrm{der}}}$ is simply the projection.

Note that $\hat{t}$ is the image of $(1, \hat{t}) \in \widehat{T}$. It comes from $\widehat{T}^\Gamma/\mathbb{C}^\times$ if and only if $(1, \hat{t})$ (or any other preimage) is Γ-invariant. For all $\sigma \in \Gamma$, Lemma A.10 and the description (A-11) lead to

$$\sigma \cdot (1, \hat{t}) = ((-1)^{A(a_1, \ldots, a_t; \sigma)}, \hat{t}).$$

The assertion follows at once. $\qquad\square$

To illustrate the use of Proposition A.11, we prove the following

Proposition A.12. *If $t = 1$ and g is odd, then $H^1(\Gamma, X^\star(T)) \cong \pi_0(\widehat{T}^\Gamma)$ is trivial.*

Proof. It suffices to show $A(-1, \ldots, -1; c) = |\Phi(c)| \notin 2\mathbb{Z}$, where $c \in \Gamma$ is the complex conjugation. Indeed, $c\phi = \phi\iota$ for all $\phi \in \Phi$ by generalities on CM-fields, so $\Phi(c) = \Phi$ has g elements, which is odd. $\quad\square$

Note that K is not assumed to be Galois over $\mathbb{Q}$.

Kottwitz's theory also relates Tate–Shafarevich groups to similar objects attached to dual tori; see Section 4 of [Kottwitz 1984b]. Nevertheless, we are not yet able to determine the Tate–Shafarevich group of T by this approach in the non-Galois case.

References

[Achter and Gordon 2017] J. D. Achter and J. Gordon, "Elliptic curves, random matrices and orbital integrals", *Pacific J. Math.* **286**:1 (2017), 1–24. With an appendix by S. A. Altuğ. MR Zbl

[Achter and Williams 2015] J. Achter and C. Williams, "Local heuristics and an exact formula for abelian surfaces over finite fields", *Canad. Math. Bull.* **58**:4 (2015), 673–691. MR Zbl

[Achter and Wong 2013] J. D. Achter and S. Wong, "Quotients of elliptic curves over finite fields", *Int. J. Number Theory* **9**:6 (2013), 1395–1412. MR Zbl

[Bitan 2011] R. A. Bitan, "The discriminant of an algebraic torus", *J. Number Theory* **131**:9 (2011), 1657–1671. MR Zbl

[Bourbaki 1985] N. Bourbaki, *Éléments de mathématique*: *Algèbre commutative*, *Chapitres 5 à 7*, Masson, Paris, 1985. MR Zbl

[Clozel 1990] L. Clozel, "The fundamental lemma for stable base change", *Duke Math. J.* **61**:1 (1990), 255–302. MR Zbl

[Conrad 2014] B. Conrad, "Reductive group schemes", pp. 93–444 in *Autour des schémas en groupes*, *I*, Panor. Synthèses **42/43**, Soc. Math. France, Paris, 2014. MR Zbl

[Cortella 1997] A. Cortella, "The Hasse principle for the similarities of bilinear forms", *Algebra i Analiz* **9**:4 (1997), 98–118. MR Zbl

[Deuring 1941] M. Deuring, "Die Typen der Multiplikatorenringe elliptischer Funktionenkörper", *Abh. Math. Sem. Hansischen Univ.* **14** (1941), 197–272. MR Zbl

[Frenkel et al. 2010] E. Frenkel, R. Langlands, and B. C. Ngô, "Formule des traces et fonctorialité: le début d'un programme", *Ann. Sci. Math. Québec* **34**:2 (2010), 199–243. MR Zbl

[Gan and Yu 2000] W. T. Gan and J.-K. Yu, "Group schemes and local densities", *Duke Math. J.* **105**:3 (2000), 497–524. MR Zbl

[Gekeler 2003] E.-U. Gekeler, "Frobenius distributions of elliptic curves over finite prime fields", *Int. Math. Res. Not.* **2003**:37 (2003), 1999–2018. MR Zbl

[Gerhard and Williams 2019] J. Gerhard and C. Williams, "Local heuristics and an exact formula for abelian varieties of odd prime dimension over finite fields", *New York J. Math.* **25** (2019), 123–144. MR Zbl

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge Univ. Press, 2006. MR Zbl

[Gordon 2022] J. Gordon, "Orbital integrals and normalization of measures", preprint, 2022. To appear in proceedings of *On the Langlands program*: *endoscopy and beyond*. arXiv 2205.02391

[Gross 1997] B. H. Gross, "On the motive of a reductive group", *Invent. Math.* **130**:2 (1997), 287–313. MR Zbl

[Gross 1998] B. H. Gross, "On the Satake isomorphism", pp. 223–237 in *Galois representations in arithmetic algebraic geometry* (Durham, 1996), edited by A. J. Scholl and R. L. Taylor, London Math. Soc. Lecture Note Ser. **254**, Cambridge Univ. Press, 1998. MR Zbl

[Gross and Gan 1999] B. H. Gross and W. T. Gan, "Haar measure and the Artin conductor", *Trans. Amer. Math. Soc.* **351**:4 (1999), 1691–1704. MR Zbl

[Guo et al. 2022] J.-W. Guo, N.-H. Sheu, and C.-F. Yu, "Class numbers of CM algebraic tori, CM abelian varieties and components of unitary Shimura varieties", *Nagoya Math. J.* **245** (2022), 74–99. MR Zbl

[Howe 2022] E. W. Howe, "Variations in the distribution of principally polarized abelian varieties among isogeny classes", *Ann. H. Lebesgue* **5** (2022), 677–702. MR Zbl

[Kisin 2017] M. Kisin, "mod p points on Shimura varieties of abelian type", *J. Amer. Math. Soc.* **30**:3 (2017), 819–914. MR Zbl

[Kottwitz 1982] R. E. Kottwitz, "Rational conjugacy classes in reductive groups", *Duke Math. J.* **49**:4 (1982), 785–806. MR Zbl

[Kottwitz 1984a] R. E. Kottwitz, "Shimura varieties and twisted orbital integrals", *Math. Ann.* **269**:3 (1984), 287–300. MR Zbl

[Kottwitz 1984b] R. E. Kottwitz, "Stable trace formula: cuspidal tempered terms", *Duke Math. J.* **51**:3 (1984), 611–650. MR Zbl

[Kottwitz 1990] R. E. Kottwitz, "Shimura varieties and λ-adic representations", pp. 161–209 in *Automorphic forms*, *Shimura varieties*, *and L-functions*, *I* (Ann Arbor, MI, 1988), edited by L. Clozel and J. S. Milne, Perspect. Math. **10**, Academic Press, Boston, 1990. MR Zbl

[Kottwitz 1992] R. E. Kottwitz, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5**:2 (1992), 373–444. MR Zbl

[Kottwitz 2005] R. E. Kottwitz, "Harmonic analysis on reductive p-adic groups and Lie algebras", pp. 393–522 in *Harmonic analysis*, *the trace formula*, *and Shimura varieties*, edited by L. Clozel and J. S. Milne, Clay Math. Proc. **4**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[Langlands 1973] R. P. Langlands, "Modular forms and ℓ-adic representations", pp. 361–500 in *Modular functions of one variable*, *II* (Proc. Internat. Summer School, Univ. Antwerp, 1972), edited by P. Deligne and W. Kuyk, Lecture Notes in Math. **349**, Springer, 1973. MR Zbl

[Liang et al. 2021] P.-X. Liang, Y. Oki, H.-Y. Yang, and C.-F. Yu, "On Tamagawa numbers of CM tori", preprint, 2021. arXiv 2109.04121

[Marseglia 2021] S. Marseglia, "Computing square-free polarized abelian varieties over finite fields", *Math. Comp.* **90**:328 (2021), 953–971. MR Zbl

[Oesterlé 1982] J. Oesterlé, "Réduction modulo p^n des sous-ensembles analytiques fermés de $\mathbb{Z}_p^N$", *Invent. Math.* **66**:2 (1982), 325–341. MR Zbl

[Ono 1961] T. Ono, "Arithmetic of algebraic tori", *Ann. of Math.* (2) **74** (1961), 101–139. MR Zbl

[Ono 1963] T. Ono, "On the Tamagawa number of algebraic tori", *Ann. of Math.* (2) **78** (1963), 47–73. MR Zbl

[Platonov and Rapinchuk 1994] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics **139**, Academic Press, Boston, 1994. MR Zbl

[Rotman 1995] J. J. Rotman, *An introduction to the theory of groups*, 4th ed., Graduate Texts in Mathematics **148**, Springer, 1995. MR Zbl

[Rüd 2022a] T. Rüd, *Tamagawa numbers of symplectic algebraic tori, orbital integrals, and mass formulae for isogeny class of abelian varieties over finite fields*, Ph.D. thesis, University of British Columbia, 2022, available at https://open.library.ubc.ca/collections/ubctheses/24/items/1.0416617.

[Rüd 2022b] T. Rüd, "Explicit Tamagawa numbers for certain algebraic tori over number fields", *Math. Comp.* **91**:338 (2022), 2867–2904. MR Zbl

[Serre 1981] J.-P. Serre, "Quelques applications du théorème de densité de Chebotarev", *Inst. Hautes Études Sci. Publ. Math.* **54** (1981), 323–401. MR Zbl

[Tate 1966] J. Tate, "Endomorphisms of abelian varieties over finite fields", *Invent. Math.* **2** (1966), 134–144. MR Zbl

[Veys 1992] W. Veys, "Reduction modulo p^n of p-adic subanalytic sets", *Math. Proc. Cambridge Philos. Soc.* **112**:3 (1992), 483–486. MR Zbl

[Voskresenskiĭ 1998] V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs **179**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Waterhouse 1969] W. C. Waterhouse, "Abelian varieties over finite fields", *Ann. Sci. École Norm. Sup.* (4) **2** (1969), 521–560. MR Zbl

[Weil 1982] A. Weil, *Adeles and algebraic groups*, Progress in Mathematics **23**, Birkhäuser, Boston, 1982. MR Zbl

[Xue and Yu 2021] J. W. Xue and C. F. Yu, "On counting certain abelian varieties over finite fields", *Acta Math. Sin.* (*Engl. Ser.*) **37**:1 (2021), 205–228. MR Zbl

[Yu 2012] C.-F. Yu, "Superspecial abelian varieties over finite prime fields", *J. Pure Appl. Algebra* **216**:6 (2012), 1418–1427. MR Zbl

j.achter@colostate.edu   *Department of Mathematics, Colorado State University, Fort Collins, CO, United States*

ali.altug@gmail.com   *New York, NY, United States*

l.e.garcia@ucl.ac.uk   *Department of Mathematics, University College London, United Kingdom*

gor@math.ubc.ca   *Department of Mathematics, University of British Columbia, Vancouver, BC, Canada*

wwli@pku.edu.cn   *Beijing International Center for Mathematical Research, Peking University, Beijing, China*

tompa.rud@gmail.com   *Massachusetts Institute of Technology, Cambridge, MA, United States*

Algebra & Number Theory

msp.org/ant

Algebra & Number Theory

Volume 17   No. 7   2023